UNIVERSITY OF CALIFORNIA
SANTA CRUZ

APPLIED DYNAMICAL SYSTEMS

AMS 214

Second-order Time-delayed Traffic Flow Through a Bottle-neck

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December 8, 2018

Abstract— **We develop a second-order time-delayed and spatially offset follow-the-leader model. The velocity of the lead vehicle is reduced as a function of position to simulate passage through a bottle-neck in the road. We discuss conditions in our model that produces both forward and back-propagating waves. To quantify our results, we examine the average velocities of each vehicle and determine whether any collisions between vehicles occur.**

I. INTRODUCTION

Congestion is a daily problem for millions of commuters. Often, after creeping along bumper-to-bumper, a traffic jam will suddenly clear and there will no sign of what caused the slow-down. In order to understand this phenomenon, many attempts have been made to mathematically model the dynamics of vehicles in traffic.

There are two major categories of traffic flow modeling: macro- and microscopic. A microscopic model tracks the dynamics of a discrete number of individual vehicles as the move along a road. In contrast, a macroscopic model 'zooms out' such that the number of vehicles goes to infinity while their lengths and spacing shrinks to zero, producing a density function. A prominent macroscopic model is the Lighthill-Whitham-Richards (LWR) model [3], [7]. It is possible to transform a first-order microscopic density-based model into the LWR macroscopic model [6]. Recent work also demonstrated that it is possible to convert a first order time-delayed microscopic model to a delayed macroscopic model that converges to LWR when the delay is taken to zero. The inclusion of a time delay is important for increasing the accuracy of follow the leader models, both in vehicular and pedestrian traffic [6]. In [4], it is shown that a flow of vehicles produces a traveling shock wave when they transition to region with a slower speed limit.

In this paper, rather than using a first order method, such as LWR, we will develop a second-order model based on [1], with the addition of a time delay and a spatial offset, and we will introduce a bottle-neck function that decreases the velocity of the lead vehicle as a function of position. We discuss conditions in our model that produces both forward and back-propagating waves and causes a collision and, to quantify our results, we will examine the average velocities of each vehicles. In some models, the non-delayed LWR in particular, collisions are impossible because the vehicles instantaneously slow to the appropriate velocity as they approach the preceding vehicle. In our model, however, collisions are possible, due to the time delay, so we include collision detection in our model.

II. MODEL

A. Setup

Our system consists of a long single-lane road which holds a series of n vehicles located at $x_1, x_2, \ldots, x_n$, where $x_{i-1} \geq x_i$ for $i = 1, \ldots, n$. We will refer to the vehicle at x_1 as the *lead vehicle*. We will also define d_i as the distance from the i^{th} vehicle to vehicle $i-1$, that is $d_i = x_{i-1} - x_i$. We will call d the *following distance*; it is strictly positive except in the case of collisions.

For simplicity, we treat the length of each vehicles as zero, therefore $x_{i-1} = x_i$ implies that vehicles $i-1$ and i are at the same location, therefore a collision has occurred.

B. Velocity Function

A key component to the modeling of traffic is the *velocity function* used to determine the motion of each vehicle. In [1] a second-order model is used, motivated by the fact that real drivers cannot directly control their velocity, only their acceleration. Therefore, each vehicle is given a *target velocity* that varies over time as a function of the following distance

$$\Lambda_i = \tanh(d_i). \tag{1}$$

The value of Λ is then used to define acceleration as

$$\ddot{x}_i \equiv \sigma \cdot (\Lambda_i - \dot{x}_i), \tag{2}$$

where σ is a *sensitivity coefficient*. The authors of [1] found that their system exhibits unstable, chaotic behavior that mimics real traffic. Their model works under the assumption, however, that vehicles instantaneously respond to the motion of the previous vehicle.

In the current work, we will modify (1) in three ways. First, we add a time-delay to imitate the reaction time of human drivers. Second, we include a spatial offset to model a safe following distance, And, third, we scale by a speed limit to create reasonable numeric results.

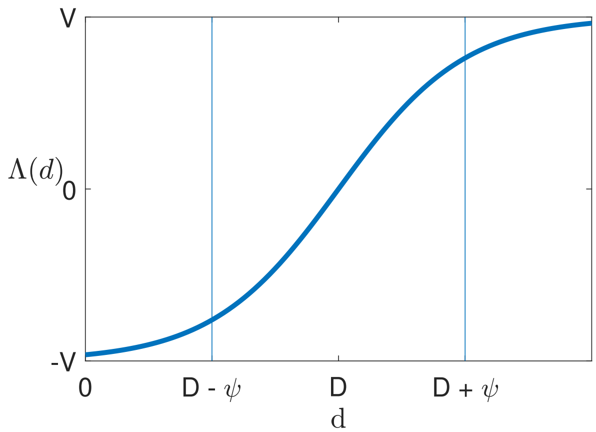

Fig. 1: Target velocity Λ as a function of following distance d.

The spatial offset is motivated by the fact that drivers do not set their velocity with the intent of achieving zero following distance. On the contrary, drivers try to achieve a safe gap between them and the previous vehicle to allow themselves to stop in time in the case of an emergency. We call this *target following distance*, D. We will treat D as a constant parameter, though in real life it would vary as a function of velocity (e.g. drivers follow more closely when driving $5\,m/s$ than $30\,m/s$).

Now, the modified velocity function becomes

$$\Lambda(d) = V \cdot \tanh\left(\frac{2}{\psi}(d-D)\right).$$

The factor of $\frac{2}{\psi}$, where ψ is the *saturation distance*, is introduced to spatially scale the width of the function around $(d-D)$. When $(d-D)=\psi$, then $\Lambda = V \cdot \tanh(2) \approx V$. We will use $\psi = D$ so that the value of Λ_i is nearly minimized around the position of x_{i-1}.

$$\Lambda(d) = V \cdot \tanh\left(\frac{2}{D}(d-D)\right) \tag{3}$$

Note that (3) reduces to the (1) when $D=0$, $V=1$, and $\psi=2$.

Now, we will add one more modification, a time delay T to model the reaction time of drivers. In our velocity function, we replace d, which is really $d(t)$, with

$$\widetilde{d}(t) \equiv d(t-T),$$

producing

$$\Lambda(\widetilde{d}) = V \cdot \tanh\left(\widetilde{d}-D\right) \tag{4}$$

We must also modify (2) to use a time-delayed version of $\dot{x}_i(t)$ with

$$\dot{\widetilde{x}}(t) \equiv \dot{x}(t-T)$$

So our time-delayed acceleration is

$$\ddot{x}(t) \equiv \sigma \cdot \left(\Lambda(\widetilde{d}) - \dot{\widetilde{x}}_i\right).$$

That is, the target velocity is based on the driver's delayed perception of the systems dynamics that happened at a past time. (See [8] for consideration of a system that uses separate delays for perception of distance and for relative velocity; the authors there show that the stability of traffic flow increases when the delay in responding to changes in position and changes in velocities are similar.)

C. Initial and history conditions

Next, we need to choose initial conditions for our system. We desire that the system starts in a steady (possibly unstable) state with the vehicles evenly distributed at distances of d_0 and moving at a constant speed v_0. The positions of the vehicles at $t=0$, therefore, are given as

$$x_i(0) = -d_0 \cdot (i-1), \quad i = 1, \dots, n.$$

Note that $x_1(0)=0$ and the starting position of all vehicles are non-positive.

Because we are using a delayed differential equation, it is insufficient to only state instantaneous initial conditions, rather it is necessary to define an *initial function*, $H : [-T, 0] \to \mathbb{R}^n$, [2]. There are two conditions on H:

- $x_{0i} = H(0)_i$
- $v_0 = \frac{d}{dt}\left(H(t)_i\right)\Big|_{t=0}$.

We note that there is a not a unique history function that satisfies these boundary conditions. Clearly, we can vary the values of $H(t)$ for values $-T \leq t < 0$, without affecting the boundary conditions. Any such function, however, is sufficient for our needs, and we find that a reasonable choice of history function is a steady state traffic flow where each vehicle is traveling at a constant velocity

$$H_i(t) = x_{0i} + v_0 \cdot t.$$

This choice of history function has the additional benefit that it is a solution to the non-delayed version of our dynamical system.

D. Lead Vehicle Velocity

Note that up this point, the dynamics of the lead vehicle has not been sufficiently defined. Equation (4) depends on the distance to the previous vehicle, but in the case of the lead vehicle, there *is* no previous vehicle. A naive approach would be to merely use $d = \infty$. This choice produces a system with reasonable dynamics, as the lead car would accelerate from its initial velocity and asymptotically approach the speed limit. The other following vehicles will in turn accelerate as the lead vehicle moves away, so that they all approach the speed limit.

It is desirable, however, for our system in the absence of bottle-neck to be in a steady state without any acceleration. To this end, we provide a distinct definition for lead vehicle's velocity

$$\dot{x}_1 \equiv \Lambda(d_0) = V \cdot \tanh\left(\frac{2}{D}(d_0 - D)\right) \tag{5}$$

1) Bottle-neck: In order to examine the behavior of traffic as it passes through a bottleneck, we modify (5) as

$$\dot{x}_1(t) = \beta(x_1) \cdot \Lambda(d_0) \tag{6}$$

where the function $\beta : [0, \infty] \to [0, 1]$ reduces the effective speed limit along the length of the road. We reduce the velocity of *only* the lead vehicle in this manner in order to examine the effect of a single vehicle slowing on those behind. In particular, we are interested in whether a slow down will propagate as a wave or die out, and if it propagates, whether it will it remain where it began or move over time. For this work, we choose

$$\beta(x) = 1 - \theta \cdot e^{-(x-c)^2/w^2}, \tag{7}$$

where $c > 0$ is the center of the bottle-neck, $w > 0$ defines the width of the bottle-neck, and $\theta \in [0, 1)$ is a coefficient which defines the decrease in velocity through the bottle-neck. We require $\theta \neq 1$, otherwise the lead car would asymptotically approach, but never reach, the bottle-neck, bringing the traffic to a full stop.

In order to ensure a smooth transition from the initial conditions, we require that $\dot{x}_1(0) = v_0$, which implies $\beta(0) = 1$. This requirement cannot be perfectly satisfied by (7). If we are careful in our selection of the bottle-neck parameters, however, the difference will be insignificant to our results, so we soften the requirement to $\beta(0) > 1 - \epsilon$, for some sufficiently small ϵ. We can then find a lower bound for c in terms of w.

$$c > w\sqrt{-\ln(\epsilon)} \tag{8}$$

If we choose $\epsilon = 10^{-6}$, then (8) becomes

$$c > 3.72 \cdot w$$

In this paper, we use a $w = 50$ meters, $c = 200$ meters, and $\theta = 0.5$, which clearly satisfy (8).

III. Results

To analyze the system, we consider the average velocity per vehicle after passing through the bottle-neck and whether a collision occurs.

A. Collision Detection

The most important attribute of a traffic system is whether for not any accidents occur. In order to detect if a given configuration of the system produces collisions, we must verify that two vehicles never occupy the same position, i.e.,

$$x_{i-1}(t) > x_i(t), \quad \forall t \in [0, t_{end}].$$

This condition can be be expressed as a function $\delta(x_i, x_{i+1}) = x_{i+1} - x_i$. We the pass this function as a termination condition to our DDE solver so that the integration aborts any time δ passes through zero.

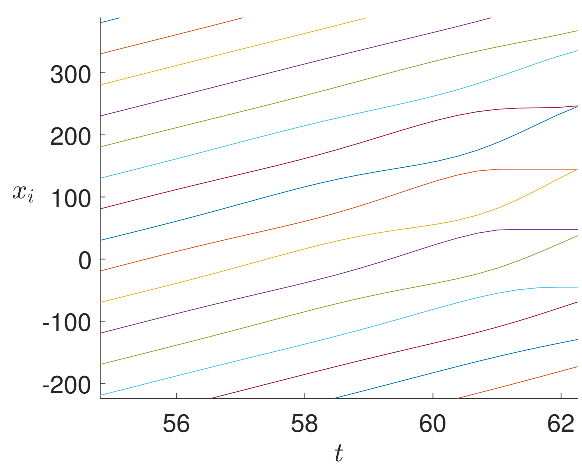

Fig. 2: A system in a steady-state demonstrates instability leading to a collision. $d_0 = 50.00\,m$, $D = 30.00\,m$, $T = 1.15\,sec$, and $\sigma = 2.00$

Using this, we can find that some systems that begin in a state of steady flow are unstable and lead to a collision. See Fig. 2 as an example of as system with a large delay time ($T = 1.15$ seconds). Small disturbances become amplified by the drivers' slow reaction. This result has obvious implications when drivers are driving at high speeds or with inhibited reaction times due to inebriation. These instabilities disappear when the delay time is decreased.

B. Average Velocities

The next value we use to evaluate our system is the average velocity of each vehicle.

$$\dot{\bar{x}} \equiv \frac{x_i(t_{end}) - x_i(0)}{t_{end}} \tag{9}$$

This is easy to compute except for one caveat: choosing the time interval. In order for the comparison of average velocities to be comparable from one vehicle to the next, and between systems, the time must start before each vehicle has entered the congestion and end after they have left it. The first is satisfied automatically, as there is no congestion at $t = 0$. The second condition is where we run into some difficulty. A traffic jam is not bounded to a specific region or interval of time. It is possible for the leading edge to travel either forward or backward, as a shock wave, with the direction depending on the flow of vehicles [5]. If the wave travels backward, against the flow of traffic, then it will always be before the center of the bottle-neck, therefore we can terminate the integration when the last vehicle passes the center. In the second case, however, where the shock wave moves forward, the point where the last vehicle has passed through is not clear, and it its certainly not before the center of the bottle-neck. There is not a distinct end to the jam, either spatially or temporally, as it gradually dissipates (See Fig. 3). Therefore, we will consider the first case and leave the second for later research.

Examination of the average velocities in a systems with a backward propagating wave produces a surprising but retrospectively intuitive result: regardless of stop-and-go traffic prior to the bottle neck, the average velocity of all vehicles are nearly equal by the time the last one passes the bottle-neck 5. The explanation for this phenomenon is

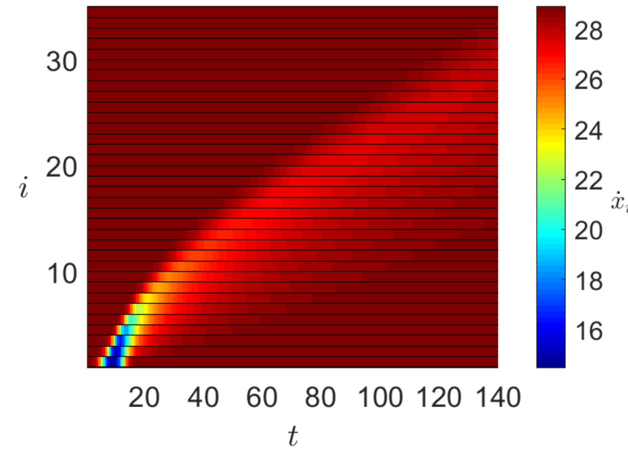

Fig. 3: Velocity $\dot{x}_i$ for each vehicle i as a function of time t. The shock-wave of a forward-moving traffic wave dissipates over time and distance. $d_0 = 80.00\,m$, $D = 40.00\,m$, $T = 0.15\,sec$, and $\sigma = 10.00$

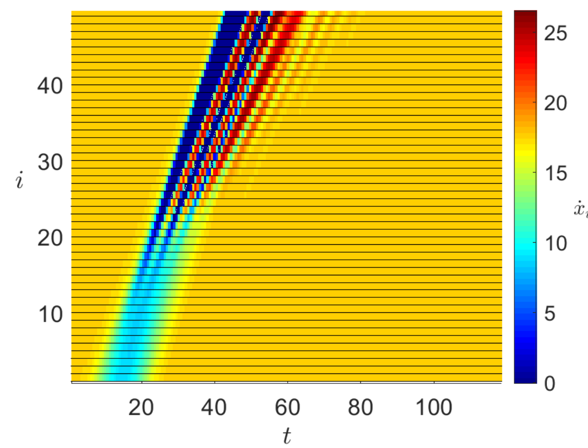

Fig. 4: A backward-propagating wave intensifies over time, leading to stop-and-go traffic. $d_0 = 40.00\,m$, $D = 30.00\,m$, $T = 0.15\,sec$, and $\sigma = 2.00$

that stop-and-go behavior materializes as vehicles approach the preceding vehicle too closely and must quickly decelerate. But, ultimately, this chaotic behavior does not matter, by the time the each vehicle passes the bottle-neck, the vehicle behind is close to the target following distance, so its speed passing through is similar to all the other vehicles. Once each vehicle is past the bottle-neck, they accelerate as necessary to restore a following distance of D, which means their final distribution closely matches their initial. The small uptick for

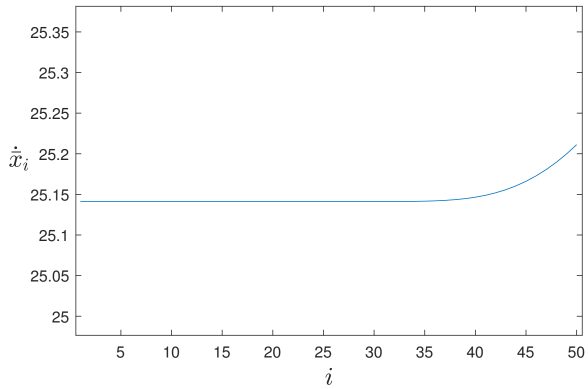

Fig. 5: Average velocity, $\dot{\bar{x}}$, for the i^{th} vehicle passing through a backward propagating traffic wave. Note the small vertical scale.

the last several vehicles, is a result of the fact that they have not yet had time to fully return to their target following distance (but they are still nearly at D)..

IV. SUMMARY

Using a second-order time-delayed system, and a target following distance, we have been able to model the dynamics of traffic. Our model was able to replicate both forward and back-propagating waves, and exhibited expected instability when long reaction times were introduced.

There are a number of ways that the model can be improved. The first would be to place limits on the range of allowable accelerations, in order to prevent accelerations that are outside the capacity of real-life automobiles. Another improvement would be to modify 4 so that the target velocity not only takes into account the current following distance, but also accounts for the velocity of the preceding vehicle.

Further investigation into the characteristics of this model, could also prove fruitful. Calculating the work done by each vehicle in the system

$$E_i = \int \max(\ddot{x}_i, 0) dx_i, \tag{10}$$

could provide insight into the relative efficiency of various values of V, D, and σ.

Finally, it would be worthwhile to consider the effects of introducing autonomous vehicles to the system. By providing a robotic driver with advanced knowledge of upcoming congestion, it could manipulate its speed (and the speed of following cars), in order to dissipate shock-waves and increase the efficiency of the entire roadway.

ACKNOWLEDGMENTS

The author thanks Tenavi Nakamura-Zimmerer and Professor Daniele Venturi for their guidance and help on this work.

REFERENCES

[1] M. Bando, K. Hasebe, A. Nakayama, A. Shibata, and Y. Sugiyama. Dynamical model of traffic congestion and numerical simulation. *Physical review. E, Statistical physics, plasmas, fluids, and related interdisciplinary topics*, 51(2):1035–1042, February 1995.

[2] R. D. Driver. Introduction to Delay Differential Equations. In R. D. Driver, editor, *Ordinary and Delay Differential Equations*, Applied Mathematical Sciences, pages 225–267. Springer New York, New York, NY, 1977.

[3] M. J. Lighthill and G. B. Whitham. On kinetic waves: II) A theory of traffic Flow on long crowded roads. In *Proc. Royal Society A229*, pages 281–345, 1955.

[4] Wen Shen. Traveling wave profiles for a Follow-the-Leader model for traffic flow with rough road condition, July 2018.

[5] Wen Shen and Karim Shikh-Khalil. Traveling waves for a microscopic model of traffic flow, March 2018.

[6] Antoine Tordeux, Guillaume Costeseque, Michael Herty, and Armin Seyfried. From traffic and pedestrian follow-the-leader models with reaction time to first order convection-diffusion flow models. *SIAM Journal of Applied Mathematics*, 78(1):63–79, 2018.

[7] Martin Treiber and Arne Kesting. The LighthillWhitham-Richards Model. In Martin Treiber and Arne Kesting, editors, *Traffic Flow Dynamics: Data, Models and Simulation*, pages 81–126. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.

[8] Lei Yu, Zhong-ke Shi, and Tong Li. A new car-following model with two delays. *Physics Letters A*, 378(4):348–357, January 2014.