Relaxed Lyapunov Conditions for Compact Sets in Dynamical Systems

Paul K. Wintz Ricardo G. Sanfelice

American Control Conference – July 8, 2025 – Denver, CO.

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HYBRID SYSTEMS LABORATORY

Lyapunov-like Theorems

Given: A dynamical system $\dot{x} = f(x)$ on $\mathbb{R}^n$ and a set $\mathcal{A} \in \mathbb{R}^n$.

To prove $\mathcal{A}$ globally asymptotically stable, construct a function $V : \mathbb{R}^n \to [0, \infty)$ such that

1. V is positive definite with respect to $\mathcal{A}$.
2. For each solution $t \mapsto x(t)$, $V(x(t))$ decreases while $x(t) \notin \mathcal{A}$.

Lyapunov-like Theorems (Smooth Systems)

Consider a set $\mathcal{A} \subset \mathbb{R}^n$, a differentiable function $V : \mathbb{R}^n \to [0, \infty)$ and

$$\dot{x} = f(x),$$

we write the rate of change of V at x as

$$\dot{V}(x) := \mathcal{L}_f V(x) := \langle \nabla V(x),\ f(x) \rangle.$$

If f is locally Lipschitz, then, the *Lyapunov conditions* are

1. For all $x \in \mathcal{A}$,
$$V(x) = 0 \quad \text{and} \quad \dot{V}(x) = 0.$$
2. For all $x \notin \mathcal{A}$,
$$V(x) > 0 \quad \text{and} \quad \dot{V}(x) < 0.$$

Lyapunov-like Theorems (Non-smooth Systems)

We consider a more general setting that allows for

1. V non-differentiable and
2. nonsmooth dynamics
   - differential inclusions: $\dot{x} \in F(x)$
   - difference inclusions: $x^+ \in G(x)$
   - hybrid systems: $\mathcal{H} : \begin{cases} \dot{x} \in F(x) & \forall x \in C \\ x^+ \in G(x) & \forall x \in D. \end{cases}$

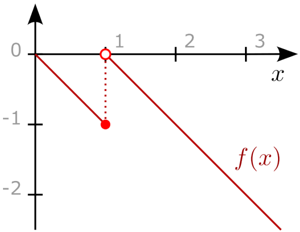
For non-smooth systems,

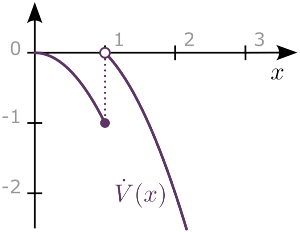
$$\left(\dot{V}(x) < 0 \text{ for all } x \notin \mathcal{A}\right) \not\Longrightarrow \left(\mathcal{A} \text{ is globally asymptotically stable}\right)$$

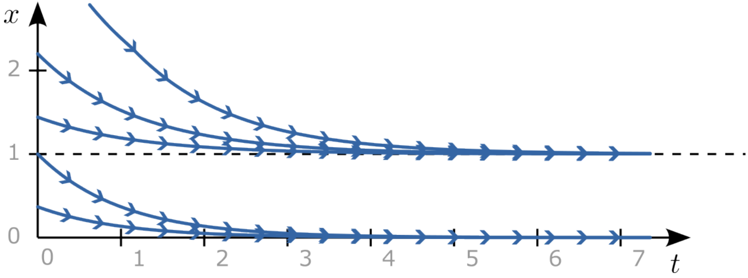
Example: $\dot{V} < 0$ without convergence to 0

$$\dot{x} = f(x) := \begin{cases} 1 - x & \text{if } x > 1 \\ -x & \text{if } x \le 1 \end{cases}$$

$$V(x) := x^2$$





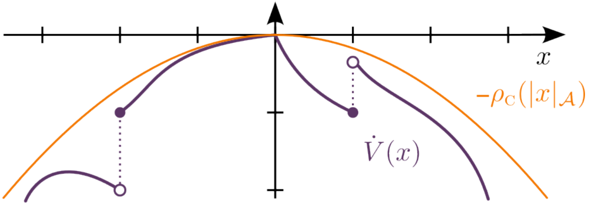


Lyapunov-like Theorems (Discontinuous Systems)

To ensure convergence despite discontinuities, we replace "$\dot{V}(x) < 0$" with

$$\dot{V}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n,$$

where $\rho_{\mathrm{C}} : [0, \infty) \to [0, \infty)$ is continuous and positive definite (w.r.t. 0).



Lyapunov-like theorems – Relaxed decrease conditions

It can be difficult to construct the function ρ_{C} to satisfy

$$\dot{V}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \quad \forall x \in \mathbb{R}^n, \tag{1}$$

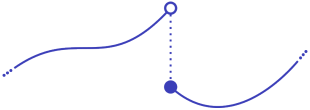
In this work, we show that if $\mathcal{A}$ is compact, then we can instead construct a lower semicontinuous function $\sigma_{\mathrm{LSC}} : \mathbb{R}^n \to [0, \infty)$

$$\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x) \quad \forall x \in \mathbb{R}^n, \tag{2}$$

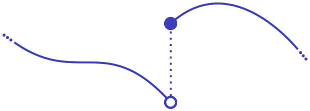
Definition (Lower Semicontinuous)

A function f is *lower semicontinuous* (LSC) at x_0 if

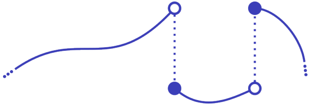
$$\liminf_{x \to x_0} f(x) \geq f(x_0).$$



Lower Semicontinuous (LSC).



Upper Semicontinuous (USC).



Not LSC, not USC.

Remark. For any function $f : \mathbb{R}^n \to \mathbb{R}$, the function

$$x \mapsto \liminf_{x' \to x} f(x')$$

is lower semicontinuous and

$$\liminf_{x' \to x} f(x') \leq f(x) \qquad \forall x \in \operatorname{dom} f$$

Thus, if we pick

$$x_0 \mapsto \sigma_{\mathrm{LSC}}(x_0) := \liminf_{x \to x_0} \Big(-\dot{V}(x) \Big),$$

then

$$\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x_0).$$

Problem Statement: Construction of ρ_{C}

Given:

- $\mathcal{A} \subset \mathbb{R}^n$ compact
- $\sigma_{\mathrm{LSC}} : \mathbb{R}^n \to [0, \infty)$ that is
  1. lower semicontinuous
  2. positive definite (w.r.t. $\mathcal{A}$).

Goal: Construct $\rho_{\mathrm{C}} : [0, \infty) \to [0, \infty)$ that is

- continuous
- positive definite (w.r.t. 0)

such that

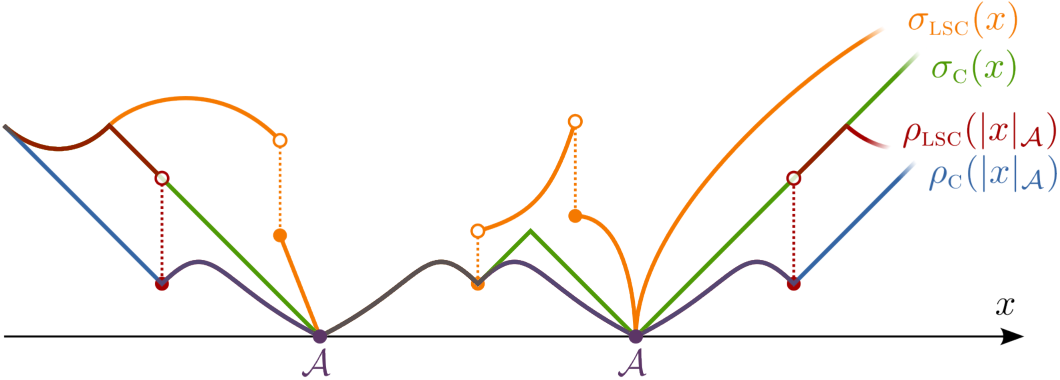
- $\rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \leq \sigma_{\mathrm{LSC}}(x)$ for all $x \in \mathbb{R}^n$.

Then, for all $x \in \mathbb{R}^n$,

$$\left(\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x)\right) \implies \left(\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}})\right).$$

Construction of ρ_{C} Outline

$$\sigma_{\mathrm{LSC}}(x) \geq \underset{\text{continuous}}{\sigma_{\mathrm{C}}(x)} \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \geq \underset{\text{continuous}}{\rho_{\mathrm{C}}(|x|_{\mathcal{A}})} > 0 \qquad \forall x \notin \mathcal{A}.$$



Lemma ($\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)$).

Given $\sigma_{\mathrm{LSC}} : \mathbb{R}^n \to [0, \infty)$

- lower semicontinuous
- positive definite w.r.t. $\mathcal{A}$

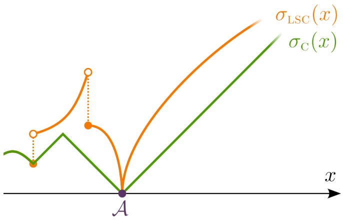
Let

$$\sigma_{\mathrm{C}}(x) := \inf_{x' \in \mathbb{R}^n} \left(\ell |x' - x| + \sigma_{\mathrm{LSC}}(x') \right) \quad \forall x \in \mathbb{R}^n.$$

Then,

- σ_{C} is Lipschitz continuous with Lipschitz constant ℓ
- σ_{C} is positive definite w.r.t. $\mathcal{A}$
- $\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)$ for all $x \in \mathbb{R}^n$.

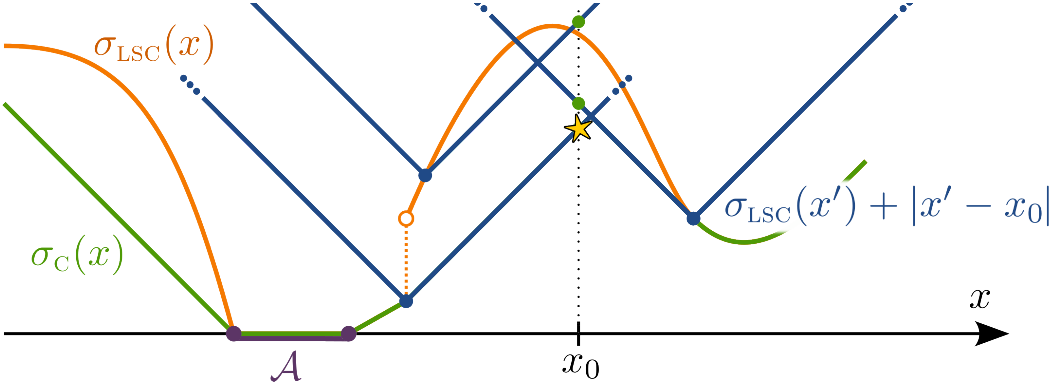
$\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)$



Construction of σ_{C}

Given $x_0 \in \operatorname{dom} \sigma_{\mathrm{LSC}}$,

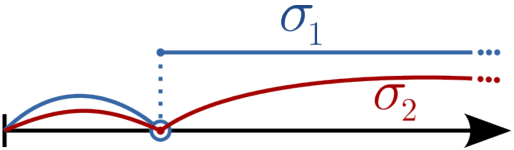
$$\sigma_{\mathrm{C}}(x_0) := \inf_{x'} \big(\sigma_{\mathrm{LSC}}(x') + |x' - x_0|\big).$$



Non-Example

Consider

$$\sigma_1(x) := \begin{cases} x(1-x) & \text{if } x \in [0,1) \\ 1 & \text{if } x \geq 1 \end{cases}$$

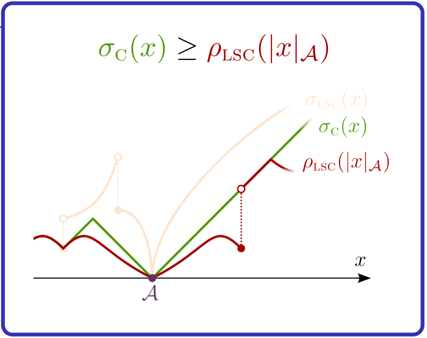


- ✅ Positive definite (w.r.t. 0)
- ❌ Not lower semicontinuous at $x = 1$.

Any continuous function σ_2 between 0 and σ_1 is 0 at $x = 1$ because

$$\liminf_{x \to 1} \sigma_1(x) = 0,$$

Lemma ($\sigma_{\mathrm{C}}(x) \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}})$).



Given $\sigma_{\mathrm{C}} : \mathbb{R}^n \to [0, \infty)$

- continuous
- positive definite w.r.t. $\mathcal{A}$

Let

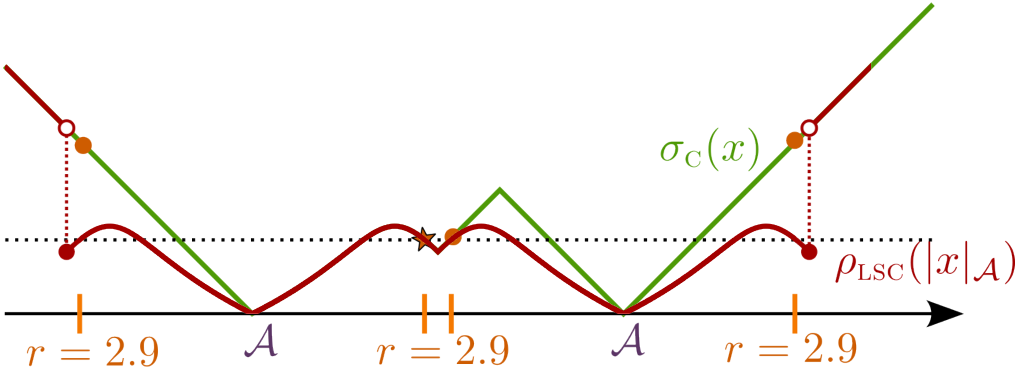
$$\rho_{\mathrm{LSC}}(r) := \inf\{\sigma_{\mathrm{C}}(x) : |x|_{\mathcal{A}} = r\} \quad \forall r \geq 0.$$

Then,

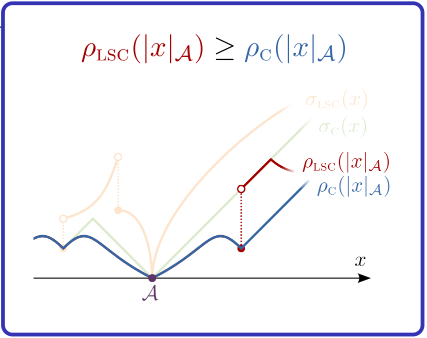
- ρ_{LSC} is lower semicontinuous
- ρ_{LSC} is positive definite (w.r.t. 0)
- $\sigma_{\mathrm{C}}(x) \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.

Construction of σ_{C}

$$\rho_{\mathrm{LSC}}(r) := \inf\{\sigma_{\mathrm{C}}(x) : |x|_{\mathcal{A}} = r\} \quad \forall r \geq 0.$$



Lemma ($\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$)



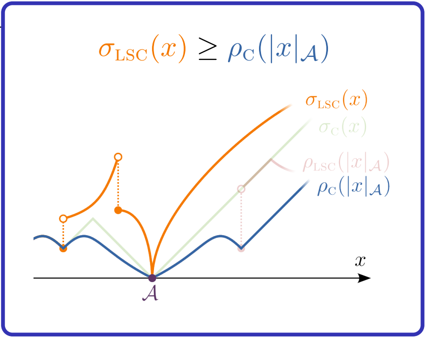
Given $\rho_{\mathrm{LSC}} : [0, \infty) \to [0, \infty)$

- lower semicontinuous,
- positive definite w.r.t. 0.

There exists ρ_{C} such that

- ρ_{C} is Lipschitz continuous
- ρ_{C} is positive definite w.r.t. 0
- $\rho_{\mathrm{LSC}}(s) \geq \rho_{\mathrm{C}}(s)$ for all $s \geq 0$.

Proposition ($\sigma_{\mathrm{LSC}}(x) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$)



Given $\sigma_{\mathrm{LSC}} : \mathbb{R}^n \to [0, \infty)$

- lower semicontinuous,
- positive definite w.r.t. $\mathcal{A}$.

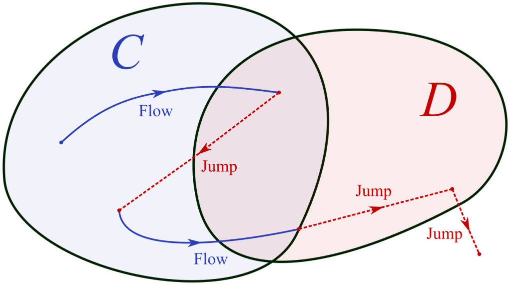
For any $\ell > 0$, there exists ρ_{C} such that

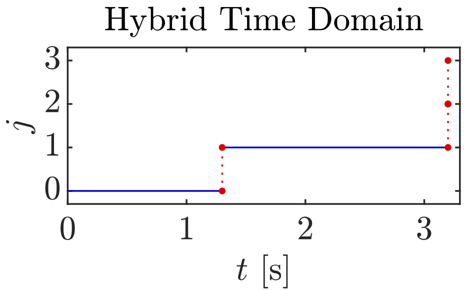
- ρ_{C} is ℓ-Lipschitz continuous,
- ρ_{C} is positive definite w.r.t. 0, and
- $\sigma_{\mathrm{LSC}}(x) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.

Hybrid Dynamical Systems

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases}$$

- flow set $C \subset \mathbb{R}^n$
- flow map $F : C \rightrightarrows \mathbb{R}^n$
- jump set $D \subset \mathbb{R}^n$
- jump map $G : D \rightrightarrows \mathbb{R}^n$





Definition (Lyapunov Function Candidate).

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$ and a set $\mathcal{A} \subset \mathbb{R}^n$.

A continuous function $V : \mathbb{R}^n \to \mathbb{R}$ is a *Lyapunov function candidate with respect to* $\mathcal{A}$ *for* $\mathcal{H}$ if

1. V is positive definite with respect to $\mathcal{A}$,
2. V is locally Lipschitz on an open neighborhood of $\overline{C}$.

Rate of change of V

Consider

$$\dot{x} \in F(x) \quad x \in C \subset \mathbb{R}^n. \tag{$\star$}$$

and a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$.

Let

$$\sup \dot{V}(x) := \sup\{\langle \zeta,\ f\rangle \mid \zeta \in \underbrace{\partial^\circ V(x)}_{\text{Generalized Gradient}},\ f \in F(x) \cap \underbrace{T_C(x)}_{\text{Tangent Cone}}\} \qquad \forall x \in C,$$

For any solution ϕ to $(\star)$ and all $t_1, t_2 \in \operatorname{dom}\phi$,

$$V\big(\phi(t_2)\big) - V\big(\phi(t_1)\big) \leq \int_{t_1}^{t_2} \sup \dot{V}\big(\phi(t)\big)\, dt.$$

Relaxed Hybrid Lyapunov Theorem

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$, a nonempty compact set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$.

Suppose that there exists $\alpha \in \mathcal{K}_\infty$ such that $\alpha(|x|_\mathcal{A}) \leq V(x)$ for all $x \in \mathbb{R}^n$, and

1. there exist lower semicontinuous functions σ_c, σ_d that are positive definite w.r.t. $\mathcal{A}$ such that

$$\begin{aligned} \sup \dot{V}(x) &\leq -\sigma_c(x) && \forall x \in C \\ V(g) - V(x) &\leq -\sigma_d(x) && \forall x \in D,\ g \in G(x). \end{aligned}$$

Then, $\mathcal{A}$ is (uniformly) globally pre-asymptotically stable for $\mathcal{H}$.

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Suppose that there exists $\alpha \in \mathcal{K}_\infty$ such that $\alpha(|x|_\mathcal{A}) \leq V(x)$ for all $x \in \mathbb{R}^n$, and

2. there exists lower semicontinuous function σ_c that is positive definite w.r.t. $\mathcal{A}$ such that

$$\sup \dot{V}(x) \leq -\sigma_c(x) \qquad \forall x \in C$$
$$V(g) - V(x) \leq 0 \qquad \forall x \in D,\ g \in G(x)$$

   and, for each $r > 0$, there exist $\Delta_T > 0$ and $\Delta_J > 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_\mathcal{A} \leq r$,
   - ϕ does not jump more than Δ_J times any interval of time shorter than Δ_T.

Then, $\mathcal{A}$ is (uniformly) globally pre-asymptotically stable for $\mathcal{H}$.

Relaxed Hybrid Lyapunov Theorem

Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$, a nonempty compact set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$.

Suppose that there exists $\alpha \in \mathcal{K}_\infty$ such that $\alpha(|x|_\mathcal{A}) \leq V(x)$ for all $x \in \mathbb{R}^n$, and

3. there exists lower semicontinuous function σ_d that is positive definite w.r.t. $\mathcal{A}$ such that

$$\sup \dot{V}(x) \leq 0 \qquad \forall x \in C$$

$$V(g) - V(x) \leq -\sigma_d(x) \qquad \forall x \in D,\ g \in G(x),$$

   and, for each $r > 0$, there exist $\Delta_T > 0$ and $\Delta_J > 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_\mathcal{A} \leq r$,
   - during any interval of time longer than Δ_T, ϕ has at least Δ_J jumps.

Then, $\mathcal{A}$ is (uniformly) globally pre-asymptotically stable for $\mathcal{H}$.

Example (Continuous-time with discontinuous F)

Consider the continuous-time system

$$\dot{x} = f(x) := -\lfloor x \rfloor \quad \forall x \in \mathbb{R},$$

Let $\mathcal{A} := [0, 1]$ and $x \mapsto V(x) := |x|^2_{\mathcal{A}}$.

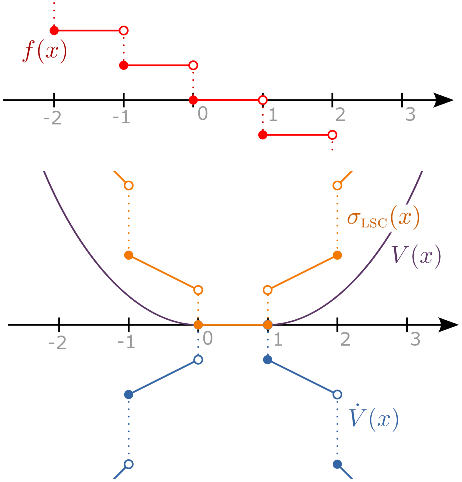
$$\dot{V}(x) = \begin{cases} -\lfloor x \rfloor |x|_{\mathcal{A}} & \text{if } x \geq 0 \\ \lfloor x \rfloor |x|_{\mathcal{A}} & \text{if } x < 0. \end{cases}$$

Let

$$\sigma_{\mathrm{LSC}}(x) := \liminf_{x' \to x} -\dot{V}(x)$$

$\implies$ $\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x)$.

$\implies$ $\mathcal{A}$ is globally asymptotically stable.



Conclusion

We presented a relaxation of the hybrid Lyapunov theorem and several insertion theorems for positive definite functions.

Future Work: Find a better upper bounded on the rate of change of V than

$$\sup \dot{V}(x) := \sup\{\langle \zeta,\ f\rangle \mid \zeta \in \partial^{\circ} V(x),\ f \in F(x) \cap T_C(x)\}$$

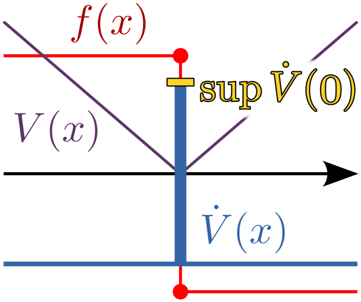
Consider

$$\dot{x} \in F(x) := \begin{cases} 1 & \text{if } x < 0 \\ [-1, 1] & \text{if } x = 0 \\ -1 & \text{if } x > 0, \end{cases}$$

$$V(x) := |x|$$

The origin is asymptotically stable...

...but $\sup(\dot{V}(0)) > 0$.



Questions?

Slides and paper available at
paulwintz.com/publications.

Funding

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Hybrid Basic Conditions

A hybrid system $\mathcal{H}$ is called *well-posed* if its set of solutions is sequentially compact, meaning that the limit of any graphically convergent sequence of solutions is also a solution. Well-posedness is useful for establishing properties such as robustness of asymptotic stability of compact sets. The following conditions are sufficient for a hybrid system to be well-posed.

Assumption 1 (Hybrid Basic Conditions Sanfelice, 2021, Def. 2.20)

A hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$ is said to satisfy the *hybrid basic conditions* if

(A1) C and D are closed;

(A2) $C \subset \operatorname{dom} F$, F is outer semicontinuous and locally bounded relative to C, and $F(x)$ is convex for each $x \in C$; and

(A3) $D \subset \operatorname{dom} G$, and G is outer semicontinuous and locally bounded relative to D.

◇

Definition (Uniform Global Asymptotic Stability).

For a continuous-time system $\dot{x} = f(x)$, a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *uniformly globally stable* for $\dot{x} = f(x)$ if there exists a class-$\mathcal{K}_\infty$ function α such that every solution ϕ to $\dot{x} = f(x)$ satisfies $|\phi(t)|_{\mathcal{A}} \leq \alpha(|\phi(0)|_{\mathcal{A}})$ for each $t \in \operatorname{dom} \phi$; and
- *uniformly globally attractive* (UGpA) for $\dot{x} = f(x)$ if for each $\varepsilon > 0$ and $r > 0$, there exists $T > 0$ such that every solution ϕ to $\dot{x} = f(x)$ with $|\phi(0)|_{\mathcal{A}} \leq r$ satisfies $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $t \in \operatorname{dom}(\phi)$ such that $t + j \geq T$.
- If $\mathcal{A}$ is both uniformly globally stable and uniformly globally pre-attractive for $\dot{x} = f(x)$, then it is said to be *uniformly globally pre-asymptotically stable* (UGpAS) for $\dot{x} = f(x)$.

The prefix "pre-" indicates that these properties allow for maximal solutions that terminate after finite time (e.g., due to leaving $C \cup D$).

Proposition (Simplified Conditions for Persistent Flows).

Consider a hybrid system $\mathcal{H}$ and a nonempty closed set $\mathcal{A}$. Suppose that for each $r \geq 0$, there exist $\Delta_T > 0$ and $\Delta_J > 0$ such that for every solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$ and for every $(t_0, j_0), (t_1, j_1) \in \operatorname{dom} \phi$,

$$|t_1 - t_0| \leq \Delta_T \implies |j_1 - j_0| \leq \Delta_J. \tag{3}$$

Then, for each $r \geq 0$, there exist $N_r \geq 0$ and $\gamma_r \in \mathcal{K}_\infty$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$t \geq \gamma_r(t + j) - N_r \quad \forall (t, j) \in \operatorname{dom} \phi. \tag{4}$$

Proposition (Simplified Conditions for Persistent Jumps).

Consider a hybrid system $\mathcal{H}$ and a nonempty closed set $\mathcal{A}$. Suppose that for each $r \geq 0$, there exists $\Delta_T > 0$ and $\Delta_J > 0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$ and for all $(t_0, j_0), (t_1, j_1) \in \operatorname{dom} \phi$,

$$|j_1 - j_0| \leq \Delta_J \implies |t_1 - t_0| \leq \Delta_T. \tag{5}$$

Then, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$j \geq \gamma_r(t + j) - N_r \quad \forall (t, j) \in \operatorname{dom} \phi. \tag{6}$$

Proof that $\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \le \sigma_{\mathrm{C}}(x)$.

For any $x \in \mathbb{R}^n$, $\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) = \inf\{\sigma_{\mathrm{C}}(x') : |x|_{\mathcal{A}} = |x'|_{\mathcal{A}}\} \le \sigma_{\mathrm{C}}(x)$. □

Proof that ρ_{LSC} is positive definite.

For each $r \ge 0$, σ_{C} attains a minimum on the compact set $\{x : |x|_{\mathcal{A}} = r\}$.

The minimum is positive if and only if $r > 0$ since σ_{C} is positive definite w.r.t. $\mathcal{A}$.

Therefore, ρ_{LSC} is positive definite (w.r.t. 0). □

Proof sketch that ρ_{LSC} is lower semicontinuous.

To establish that ρ_{LSC} is LSC, we exploit the fact that $\mathcal{A}$ is compact and σ_{C} is continuous. For each $r \ge 0$, we pick a compact set K_r containing an open neighborhood of $\mathcal{A} + r\mathbb{B}$. Since σ_{C} is continuous, its restriction to the compact set K_r is uniformly continuous. This allows us to do a δ-ε proof of lower semicontinuity. □