Relaxed Lyapunov Conditions for Compact Sets in Dynamical Systems

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***Abstract*— In the setting of continuous-time, discrete-time, and hybrid systems, including differential inclusions and difference inclusions, relaxations are given for Lyapunov functions to establish uniform global pre-asymptotic stability (UGAS) of compact sets. It is shown that for a compact set, if there exist a Lyapunov function and two lower semicontinuous functions that are positive definite with respect to the compact set and whose negations are upper bounds on the rate of change of the Lyapunov function during flows and jumps, respectively, then the compact set is UGAS. Under additional regularity conditions, conditions sufficient to show that a compact set is UGAS are further weakened to merely require the rate of change of the Lyapunov function is negative definite. Simplified conditions on hybrid time domains, compared to existing results, are given to establish that a set is UGAS for hybrid systems when the Lyapunov function is merely nonincreasing during either flows or jumps.**

I. Introduction

In several Lyapunov-like theorems found in the control theory literature, assumptions are imposed on a function $h : \mathbb{R}^n \to \mathbb{R}$ in the form

$$h(x) \leq -\rho(|x|_{\mathcal{A}}) \quad \forall x \in \mathbb{R}^n, \tag{1}$$

where $|x|_{\mathcal{A}}$ is the distance from $x \in \mathbb{R}^n$ to a set $\mathcal{A}$, and $\rho : [0, \infty) \to [0, \infty)$ is continuous and positive definite. E.g., for a system $\dot{x} = f(x)$ with a differentiable Lyapunov function V, we would use $h := \dot{V}$, where $\dot{V}(x) := \langle \nabla V(x), f(x) \rangle$. Examples of assumptions in the form (1) include the hybrid Lyapunov theorem [1, Thm. 3.19(3)], (robust) control Lyapunov functions [1, Defs. 10.2 and 10.14], and input-to-state stability (ISS) Lyapunov functions [2], [3]. In some results, namely [4, Thms. 4.1 and 4.9], assumptions are given without using the distance function in the form

$$h(x) \leq -\sigma(x) \quad \forall x \in \mathbb{R}^n, \tag{2}$$

where $\sigma : \mathbb{R}^n \to [0, \infty)$ is continuous and positive definite with respect to $\mathcal{A}$, but such existing results assume $\mathcal{A} = \{0\}$.

In this paper, we relax the assumptions on Lyapunov functions for the case where $\mathcal{A}$ is compact. This work builds upon the hybrid Lyapunov theorems [5, Thm. 3.18] and [1, Thm. 3.19]. In particular, [1, Thm. 3.19] asserts that a given set $\mathcal{A}$ is uniformly globally asymptotically stable with respect to a given hybrid system $\mathcal{H}$ under given assumptions. This paper relaxes the assumptions of [1, Thm. 3.19] by i) relaxing bounds on the rate of change of V that are given as a function of the distance from $\mathcal{A}$, as in (1), to only a function of the state, as in (2), ii) allowing for bounds to be lower semicontinuous instead of continuous, iii) relaxing the typical $\mathcal{K}_\infty$ upper-bound on V, and iv) simplifying conditions on hybrid time domains when V is merely nondecreasing during flows or across jumps. We prove our results in the context of hybrid dynamical systems, with the results for discrete-time and continuous-time systems following as special cases. Along the way, we also prove several auxiliary results relating to finding lower bounds for positive definite lower semicontinuous functions that may be useful in other contexts.

The remainder of this paper is structured as follows. Section I-A introduces notation and definitions, Section I-B introduces hybrid systems, and Section I-C defines stability properties. Section II contains insertion theorems that assert the existence of functions between constraints. Section III presents our main result, a Lyapunov theorem to show that compact sets are UGAS for hybrid systems, which relaxes results in [1], [5]. In Section III-A, simplified conditions are provided for establishing bounds on the amount of flow versus the number of jumps in a hybrid time domain. Section III-B presents corollaies of our hybrid Lyapunov theorem for the special cases of continuous-time and discrete-time systems. Due to limited space, proofs are omitted or abbreviated.

A. *Preliminaries*

Let $\mathbb{R}_{\geq 0} := [0, \infty)$ and $\mathbb{N} := \{0, 1, 2, \ldots\}$. The Euclidean norm of $x \in \mathbb{R}^n$ is written $|x|$ and the inner product between x and $y \in \mathbb{R}^n$ is written $\langle x, y \rangle$. We write the unit ball in $\mathbb{R}^n$ as $\mathbb{B} := \{x \in \mathbb{R}^n : |x| \leq 1\}$. For a set $S \subset \mathbb{R}^n$, we denote the boundary as ∂S, the interior as $\mathrm{int}(S)$, and the closure as $\overline{S}$. We write the convex hull of S as $\mathrm{conv}(S)$. If S is nonempty, then the distance from $x \in \mathbb{R}^n$ to S is $|x|_S := \inf_{y \in S} |y - x|$. If $S \subset \mathbb{R}^n$ is nonempty, then the *contingent cone* of S at $x \in \overline{S}$ is denoted $T_S(x)$ [6].

Given $f : \mathbb{R}^n \to \mathbb{R}^m$, its domain is denoted $\mathrm{dom}\, f$. If f is differentiable at $x \in \mathrm{dom}\, f$, then the gradient of f at x is denoted $\nabla f(x)$. We say f is *smooth* if it is infinitely differentiable. We say f is *lower semicontinuous* (LSC) if

$$f(x_0) \leq \liminf_{x \to x_0} f(x) \quad \forall x_0 \in \mathrm{dom}\, f.$$

If f is LSC, then $g := -f$ is *upper semicontinuous* (USC).

The domain of a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^m$ is written $\mathrm{dom}\, F := \{x \in \mathbb{R}^n \mid F(x) \neq \varnothing\}$. We say F is *outer semicontinuous* (OSC) [1, Def. A.32] if for each $x_0 \in \mathrm{dom}\, F$,

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each sequence $\{x_i\}_{i=1}^{\infty}$ in $\operatorname{dom} F$ converging to x_0, and each convergent sequence $\{y_i\}_{i=1}^{\infty}$ with each $y_i \in F(x_i)$, we have that $\lim_{i\to\infty} y_i \in F(x_0)$. We say F is *locally bounded* if for each $x_0 \in \operatorname{dom} F$, there exists a neighborhood U of x_0 such that $F(U \cap \operatorname{dom} F)$ is bounded [1, Def. A.11].

A continuous function $\alpha : [0, c) \to \mathbb{R}_{\geq 0}$ is class–$\mathcal{K}$ if α is zero at zero and strictly increasing. A continuous function $\alpha : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be in class–$\mathcal{K}_\infty$ if it is zero at zero, strictly increasing, and $\lim_{r\to\infty} \alpha(r) = \infty$. A function $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is *positive definite* if $\rho(0) = 0$ and $\rho(r) > 0$ for all $r > 0$. We write the set of all positive definite functions on $\mathbb{R}_{\geq 0}$ as $\mathcal{PD}(0)$. Given nonempty sets $\mathcal{A} \subset \mathbb{R}^n$ and $\mathcal{X} \subset \mathbb{R}^n$, a function $\sigma : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is said to be *positive definite on* $\mathcal{X}$ *with respect to* $\mathcal{A}$ if $\sigma(x) = 0$ for all $x \in \mathcal{A} \cap \mathcal{X}$ and $\sigma(x) > 0$ for all $x \in \mathcal{X} \setminus \mathcal{A}$. The set of all positive definite functions on $\mathcal{X} = \mathbb{R}^n$ with respect to $\mathcal{A}$ is denoted $\mathcal{PD}(\mathcal{A})$. A function f is said to be *negative definite* if $g :=$ is positive definite. Throughout this paper, we use the notation "ρ" to denote positive definite functions on $\mathbb{R}_{\geq 0}$ and "σ" to denote positive definite functions on $\mathbb{R}^n$ with respect to $\mathcal{A}$ (i.e., $\rho \in \mathcal{PD}(0)$ and $\sigma \in \mathcal{PD}(\mathcal{A})$).

For nonsmooth functions, we use the Clarke generalized gradient and Clarke generalized directional derivative [7]. For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, the *Clarke generalized gradient* of V at any $x \in \mathbb{R}^n$ is

$$\partial^\circ V(x) := \operatorname{conv}\left\{ \lim_{i\to\infty} \nabla V(x_i) \;\middle|\; \begin{array}{l} \exists (x_i \to x) \text{ s.t. } V \text{ is} \\ \text{differentiable at each } x_i \end{array} \right\}.$$

The *Clarke generalized directional derivative* of V at $x \in \mathbb{R}^n$ in the direction $w \in \mathbb{R}^n$ is given by

$$V^\circ(x, w) = \max_{\zeta \in \partial^\circ V(x)} \langle \zeta,\, w \rangle. \tag{3}$$

B. Hybrid Systems

We consider hybrid systems on $\mathbb{R}^n$, as in [1], [5],

$$\mathcal{H} : \begin{cases} \dot{x} \ \in F(x) & x \in C \\ x^+ \in G(x) & x \in D \end{cases} \tag{4}$$

with state $x \in \mathbb{R}^n$, flow set $C \subset \mathbb{R}^n$, flow map $F : C \rightrightarrows \mathbb{R}^n$, jump set $D \subset \mathbb{R}^n$, and jump map $G : D \rightrightarrows \mathbb{R}^n$. We write $\mathcal{H}$ compactly as $\mathcal{H} = (C, F, D, G)$.

A *solution* ϕ to $\mathcal{H}$ is defined on a hybrid time domain $\operatorname{dom}\phi \subset \mathbb{R}_{\geq 0} \times \mathbb{N}$, which parameterizes the solution by ordinary time $t \in \mathbb{R}_{\geq 0}$ and discrete time $j \in \mathbb{N}$. Roughly speaking, the set $\operatorname{dom}\phi$ is a *hybrid time domain* if there either exists a (finite or infinite) sequence

$$0 \leq t_1 \leq t_2 \leq \cdots \tag{5}$$

such that for every $(T, J) \in \operatorname{dom}\phi$,

$$\begin{aligned} &\operatorname{dom}\phi \cap \big([0,T]\times\{0,1,\ldots,J\}\big) \\ &\quad = \big([0,t_1]\times\{0\}\big) \cup \big([t_1,t_2]\times\{1\}\big) \cup \cdots \cup \big([t_J,T]\times\{J\}\big), \end{aligned}$$

and for all $(t,j), (t',j') \in \operatorname{dom}\phi$,

$$t \geq t' \iff j \geq j'.$$

Each t_j in (5) is called a *jump time* in $\operatorname{dom}\phi$. If the interval $I_j := \{t \mid (t,j) \in \operatorname{dom}\phi\}$ has a nonempty interior, then I_j is called an *interval of flow*. At each jump time t_j in $\operatorname{dom}\phi$, the solution ϕ must satisfy $\phi(t_j, j-1) \in D$ and

$$\phi(t_j, j) \in G\big(\phi(t_j, j-1)\big).$$

For each interval of flow I_j, ϕ must satisfy $\phi(t,j) \in C$ for all $t \in \operatorname{int} I_j$ and

$$\frac{d\phi}{dt}(t,j) \in F(\phi(t,j)) \quad \text{for almost all } t \in I_j.$$

Let $\sup_t \operatorname{dom}\phi := \sup\{t \in \mathbb{R}_{\geq 0} \mid (t,j) \in \operatorname{dom}\phi\}$ and $\sup_j \operatorname{dom}\phi := \sup\{j \in \mathbb{N} \mid (t,j) \in \operatorname{dom}\phi\}$. We say ϕ is *complete* if $\operatorname{dom}\phi$ is unbounded (namely, $\sup_t \operatorname{dom}\phi = \infty$ or $\sup_j \operatorname{dom}\phi = \infty$). A solution ϕ to $\mathcal{H}$ is said to be a *maximal solution* if there does not exist another solution ϕ' to $\mathcal{H}$ such that ϕ is a truncation of ϕ' with $\operatorname{dom}\phi$ a strict subset of $\operatorname{dom}\phi'$.

A hybrid system $\mathcal{H}$ is called *well-posed* if its set of solutions is sequentially compact, meaning that the limit of any graphically convergent sequence of solutions is also a solution. Well-posedness is useful for establishing properties such as robustness of asymptotic stability of compact sets. The following conditions are sufficient for a hybrid system to be well-posed.

Assumption 1 (Hybrid Basic Conditions [1, Def. 2.20]). A hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$ is said to satisfy the *hybrid basic conditions* if

- (A1) C and D are closed;
- (A2) $C \subset \operatorname{dom} F$, F is outer semicontinuous and locally bounded relative to C, and $F(x)$ is convex for each $x \in C$; and
- (A3) $D \subset \operatorname{dom} G$, and G is outer semicontinuous and locally bounded relative to D. ◇

C. Stability Properties

We consider uniform global pre-asymptotic stability of sets, which is a stronger condition than global pre-asymptotic stability due to requiring that for each $\varepsilon > 0$ and $r > 0$, there is a uniform bound $T > 0$ on the hybrid time it takes any hybrid solution that starts within a distance of r from $\mathcal{A}$ to converge within a distance ε from $\mathcal{A}$. The prefix "pre-" indicates that these properties allow for maximal solutions that terminate after finite time (e.g., due to leaving $C \cup D$).

Definition 1 ([1, Def. 3.7]). For a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$, a nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *uniformly globally stable* for $\mathcal{H}$ if there exists a class-$\mathcal{K}_\infty$ function α such that every solution ϕ to $\mathcal{H}$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \alpha(|\phi(0,0)|_{\mathcal{A}})$ for each $(t,j) \in \operatorname{dom}\phi$; and
- *uniformly globally pre-attractive* (UGpA) for $\mathcal{H}$ if for each $\varepsilon > 0$ and $r > 0$, there exists $T > 0$ such that every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq r$ satisfies $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom}(\phi)$ such that $t + j \geq T$.
- If $\mathcal{A}$ is both uniformly globally stable and uniformly globally pre-attractive for $\mathcal{H}$, then it is said to be *uniformly globally pre-asymptotically stable* (UGpAS) for $\mathcal{H}$. ◇

A nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be *forward pre-invariant* for $\mathcal{H}$ if each solution ϕ to $\mathcal{H}$ with $\phi(0,0) \in \mathcal{A}$ satisfies $\phi(t,j) \in \mathcal{A}$ for all $(t,j) \in \operatorname{dom}\phi$ [1, Def. 3.13].

If every maximal solution to $\mathcal{H}$ is complete, then the "pre-" prefixes are omitted, in which case, if $\mathcal{A}$ is UGpA, UGpAS, or forward pre-invariant, then we say $\mathcal{A}$ is, respectively, *uniformly globally attractive* (UGA), *uniformly globally asymptotically stable* (UGAS), or *forward invariant*.

II. Positive Definite and $\mathcal{K}_\infty$ Insertion Theorems

In the field of topology, an *insertion theorem* asserts the ability to insert a function between two other functions. An example is the Katětov–Tong insertion theorem [8], which allows for the insertion of a continuous function between any USC function $\ell : \mathbb{R} \to \mathbb{R}$ and LSC function $u : \mathbb{R} \to \mathbb{R}$ such that $\ell \leq u$. In this section, we introduce results for inserting positive definite functions between zero and another positive definite function. These results are used, in Section III, to relax conditions such as (1) and (2).

A. Positive Definite Functions

Our first result shows that given any LSC function $\sigma_{\text{LSC}} \in \mathcal{PD}(\mathcal{A})$, we can construct a Lipschitz continuous function $\sigma_{\text{C}} \in \mathcal{PD}(\mathcal{A})$ such that $\sigma_{\text{C}} \leq \sigma_{\text{LSC}}$.

Proposition 1. *Consider a closed set $\mathcal{X} \subset \mathbb{R}^n$, a compact set $\mathcal{A} \subset \mathcal{X}$, a function $\sigma_{\text{LSC}} : \mathcal{X} \to \mathbb{R}_{\geq 0}$, and any $\ell > 0$. Let $\sigma_{\text{C}} : \mathcal{X} \to \mathbb{R}_{\geq 0}$ be defined by*

$$\sigma_{\text{C}}(x) := \inf_{x^* \in \mathcal{X}} \big(\ell|x^* - x| + \sigma_{\text{LSC}}(x^*)\big) \quad \forall x \in \mathcal{X}. \tag{6}$$

If σ_{LSC} is in $\mathcal{PD}(\mathcal{A})$ and LSC, then σ_{C} is in $\mathcal{PD}(\mathcal{A})$, ℓ–Lipschitz continuous, and

$$\sigma_{\text{C}}(x) \leq \sigma_{\text{LSC}}(x) \quad \forall x \in \mathcal{X}. \tag{7}$$

Proof Sketch. For each $x_0 \in \mathcal{X}$, let

$$x \mapsto \mathcal{L}_{x_0}(x) := \ell|x - x_0| + \sigma_{\text{LSC}}(x) \quad \forall x \in \mathcal{X}. \tag{8}$$

Substituting $\mathcal{L}_{x_0}$ into (6), it can be shown that

$$\sigma_{\text{C}}(x_0) = \inf_{x \in \mathcal{X}} \mathcal{L}_{x_0}(x) = \min_{x \in K} \mathcal{L}_{x_0}(x),$$

where $K := x_0 + \sigma_{\text{LSC}}(x_0)\mathbb{B}$ is compact. Since $x_0 \in K$, we have that $\sigma_{\text{C}}(x_0) \leq \sigma_{\text{LSC}}(x_0)$.

To establish that σ_{C} is positive definite on $\mathcal{X}$ with respect to $\mathcal{A}$, we see that if $x_0 \in \mathcal{A}$, then $\sigma_{\text{C}}(x_0) = \sigma_{\text{LSC}}(x_0) = 0$. Suppose, alternatively, that $x_0 \notin \mathcal{A}$ and let $x^* \in K$ be a minimizer of $\mathcal{L}_{x_0}$. If $x_* = x_0$, then $\sigma_{\text{C}}(x_0) = \ell|x_0 - x^*| + \sigma_{\text{LSC}}(x^*) > 0$ because $\sigma_{\text{LSC}}(x^*) > 0$. Otherwise, $\sigma_{\text{C}}(x_0) = \ell|x_0 - x^*| + \sigma_{\text{LSC}}(x^*) > 0$ because $|x_0 - x^*| > 0$.

Lipschitz continuity is shown by taking any $x_0, x_1 \in \mathcal{X}$ and (as before) $x^* \in K$ that minimizes $\mathcal{L}_{x_0}(x)$. Using the fact that $\sigma_{\text{C}}(x_0) = \mathcal{L}_{x_0}(x^*)$ and $\sigma_{\text{C}}(x_0) \leq \mathcal{L}_{x_1}(x^*)$, and applying the inverse triangle inequality, we find

$$\sigma_{\text{C}}(x_1) - \sigma_{\text{C}}(x_0) \leq \mathcal{L}_{x_1}(x^*) - \mathcal{L}_{x_0}(x^*) \leq \ell|x_1 - x_0|.$$

Swapping x_0 and x_1 completes the proof. □

Example 1. To see why σ_{LSC} is assumed to be LSC in Proposition 1, consider $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ defined for all $x \geq 0$ by

$$\sigma(x) := x(1-x) \text{ if } x \in [0,1) \quad \text{and} \quad \sigma(x) := 1 \text{ if } x \geq 1.$$

Although σ is positive definite with respect to $\mathcal{A} := \{0\}$, it cannot be lower bound by a continuous function in $\mathcal{PD}(\mathcal{A})$ because σ is not LSC at $x = 1$ and $\liminf_{x \to 1} \sigma(x) = 0$. In particular, for any continuous function $\sigma_{\text{C}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that $\sigma_{\text{C}}(x) \leq \sigma(x)$, it must be that $\sigma_{\text{C}}(1) = 0$ because σ_{C} is squeezed to zero by σ as x approaches 1 from the left. ◇

The next result allows us to weaken assumptions in the form of (1), with ρ continuous, into an inequality in the form of (2) with σ LSC.

Lemma 1. *Consider a continuous function $\sigma_{\text{C}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ and a compact set $\mathcal{A} \subset \mathbb{R}^n$. If $\sigma_{\text{C}} \in \mathcal{PD}(\mathcal{A})$, then*

$$r \mapsto \rho_{\text{LSC}}(r) := \inf\big\{\sigma_{\text{C}}(x) : |x|_{\mathcal{A}} = r\big\} \quad \forall r \geq 0 \tag{9}$$

is LSC, positive definite, and satisfies

$$\rho_{\text{LSC}}(|x|_{\mathcal{A}}) \leq \sigma_{\text{C}}(x) \quad \forall x \in \mathbb{R}^n. \tag{10}$$

Proof Sketch. For each $r \geq 0$, σ_{C} attains a minimum on the compact set $\{x : |x|_{\mathcal{A}} = r\}$, and the minimum is positive if and only if $r > 0$ since $\sigma_{\text{C}} \in \mathcal{PD}(\mathcal{A})$. Thus, $\rho_{\text{LSC}} \in \mathcal{PD}(0)$.

To establish that ρ_{LSC} is LSC, we exploit the fact that $\mathcal{A}$ is compact and σ_{C} is continuous. For each $r \geq 0$, we pick a compact set K_r containing an open neighborhood of $\mathcal{A} + r\mathbb{B}$. Since σ_{C} is continuous, its restriction to the compact set K_r is uniformly continuous. This allows us to do a δ-ε proof of lower semicontinuity. □

The next example shows a case where the function ρ_{LSC} in Lemma 1 is merely LSC—not continuous.

Example 2. Consider $\mathcal{A} := \{-1, 1\} \subset \mathbb{R}$, and let $\sigma_{\text{C}}(x) := |x^2 - 1|$ for all $x \in \mathbb{R}$. Then, for all $r \geq 0$,

$$\rho_{\text{LSC}}(r) = \big\{|x^2 - 1| : |x|_{\mathcal{A}} = r\big\} = \begin{cases} r(2-r) & \text{if } r \leq 1 \\ r(2+r) & \text{if } r > 1, \end{cases}$$

so ρ_{LSC} jumps from $\rho_{\text{LSC}}(1) = 1$ to $\rho_{\text{LSC}}(1.001) > 2$. ◇

The following result asserts that for every LSC function $\sigma_{\text{LSC}} \in \mathcal{PD}(\mathcal{A})$ with $\mathcal{A}$ compact, we can construct a continuous function $\rho_{\text{C}} \in \mathcal{PD}(0)$ that—when composed with the distance from $\mathcal{A}$, as in (1)—is a lower bound on σ_{LSC}.

Proposition 2. *Consider a compact set $\mathcal{A} \subset \mathbb{R}^n$. For each LSC function $\sigma_{\text{LSC}} : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ in $\mathcal{PD}(\mathcal{A})$ and each $\ell > 0$, there exists an ℓ-Lipschitz continuous and positive definite function $\rho_{\text{C}} : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ such that*

$$\rho_{\text{C}}(|x|_{\mathcal{A}}) \leq \sigma_{\text{LSC}}(x) \quad \forall x \in \mathbb{R}^n. \tag{11}$$

Proof. Suppose $\sigma_{\text{LSC}} \in \mathcal{PD}(\mathcal{A})$ is LSC. By Proposition 1, there exists a continuous function $\sigma_{\text{C}} \in \mathcal{PD}(\mathcal{A})$ such that

$$\sigma_{\text{C}}(x) \leq \sigma_{\text{LSC}}(x) \quad \forall x \in \mathbb{R}^n.$$

By Lemma 1, there exists an LSC and positive definite function $\rho_{\text{LSC}} \in \mathcal{PD}(0)$ such that

$$\rho_{\text{LSC}}(|x|_{\mathcal{A}}) \leq \sigma_{\text{C}}(x) \quad \forall x \in \mathbb{R}^n.$$

Again, by Proposition 1, for any $\ell > 0$ there exists an ℓ-Lipschitz continuous function $\rho_{\text{C}} \in \mathcal{PD}(0)$ such that

$$\rho_{\text{C}}(r) \leq \rho_{\text{LSC}}(r) \quad \forall r \geq 0.$$

Thus, for all $x \in \mathbb{R}^n$,

$$\rho_{\text{C}}(|x|_{\mathcal{A}}) \leq \rho_{\text{LSC}}(|x|_{\mathcal{A}}) \leq \sigma_{\text{C}}(x) \leq \sigma_{\text{LSC}}(x). \qquad \square$$

B. $\mathcal{K}_\infty$ Insertion Theorems

This section shows that for any nonempty compact set $\mathcal{A}$ and continuous function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$, there exists $\alpha \in \mathcal{K}_\infty$ such that

$$V(x) \leq \alpha(|x|_{\mathcal{A}}) \quad \forall x \in \operatorname{dom} V. \tag{12}$$

Lemma 2. *Consider a closed and nonempty set $\mathcal{X} \subset \mathbb{R}^n$, a compact and nonempty set $\mathcal{A} \subset \mathcal{X}$, and a continuous function $V : \mathcal{X} \to \mathbb{R}_{\geq 0}$. If $V(x) = 0$ for all $x \in \mathcal{A}$, then there exists $\alpha \in \mathcal{K}_\infty$ such that $V(x) \leq \alpha(|x|_{\mathcal{A}})$ for all $x \in \mathcal{X}$.*

To see why the conclusion in Lemma 2 does not generally hold if $\mathcal{A}$ is unbounded, consider $\mathcal{A} := \mathbb{R} \times \{0\}$ and $(x_1, x_2) \mapsto V(x_1, x_2) = (|x_1| + 1)|x_2|$.

III. Lyapunov Theorems for Compact Sets

In this section, we present a Lyapunov theorem with relaxed assumptions for showing that a compact set is UGpAS for a hybrid system. The following definition establishes the class of functions permissible as Lyapunov functions.

Definition 2 ([1, Def. 3.17])**.** Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$ and a set $\mathcal{A} \subset \mathbb{R}^n$. A function $V : \operatorname{dom} V \subset \mathbb{R}^n \to \mathbb{R}$ is a *Lyapunov function candidate with respect to $\mathcal{A}$ for $\mathcal{H}$* if $\overline{C} \cup D \cup G(D) \subset \operatorname{dom} V$, V is positive definite on $C \cup D \cup G(D)$ with respect to $\mathcal{A}$, V is continuous, and V is locally Lipschitz on an open neighborhood of $\overline{C}$. ◇

A key part of any Lyapunov-like theorem is establishing an upper bound on the change in a Lyapunov function candidate V. For hybrid systems, the following functions provide upper bounds on the rate of V during flows and across jumps.

Definition 3. Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$, a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$. We define

$$u_{\text{C}}(x) := \sup_{f \in F(x) \cap T_C(x)} V^\circ(x, f) \qquad \forall x \in C \tag{13}$$

$$u_{\text{D}}(x) := \sup_{g \in G(x)} V(g) - V(x) \qquad \forall x \in D. \tag{14}$$

The suprema are defined as $-\infty$ if the domains are empty (e.g., if $F(x) \cap T_C(x) = \varnothing$, then $u_{\text{C}}(x) = -\infty$). Recall that $T_C(x)$ is the contingent cone of C at x. ◇

For any solution ϕ to $\mathcal{H}$ and all $(t_1, j_1), (t_2, j_2) \in \operatorname{dom} \phi$,

$$V\big(\phi(t_2, j_2)\big) - V\big(\phi(t_1, j_1)\big) \leq \int_{t_1}^{t_2} u_{\text{C}}\big(\phi(t, \underline{j}(t))\big)\, dt + \sum_{j=j_1}^{j_2-1} u_{\text{D}}\big(\phi(\underline{t}(j), j)\big),$$

where $\underline{j}$ and $\underline{t}$ are defined for all $(t, j) \in \operatorname{dom} \phi$ by

$$j \mapsto \underline{t}(j) := \min\{t' \mid (t', j) \in \operatorname{dom} \phi\}$$
$$t \mapsto \underline{j}(t) := \min\{j' \mid (t, j') \in \operatorname{dom} \phi\}.$$

The main result of this paper, which follows the structure of [1, Thm. 3.19(3)], is presented next. In particular, Theorem 1 provides sufficient conditions for a compact set to be UGpAS.

Theorem 1. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$, a nonempty compact set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$. Suppose there exists $\alpha_1 \in \mathcal{K}_\infty$ such that*

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \quad \forall x \in C \cup D \cup G(D). \tag{15}$$

Then, the set $\mathcal{A}$ is UGpAS for $\mathcal{H}$ if any of the following conditions hold:

(a) Strict decrease during flows and jumps: *There exist LSC functions $\sigma_c, \sigma_d \in \mathcal{PD}(\mathcal{A})$ such that*

$$u_{\text{C}}(x) \leq -\sigma_c(x) \qquad \forall x \in C \tag{16}$$

$$u_{\text{D}}(x) \leq -\sigma_d(x) \qquad \forall x \in D. \tag{17}$$

(b) Strict decrease during flows, nonincreasing at jumps: *There exists an LSC function $\sigma_c \in \mathcal{PD}(\mathcal{A})$ such that*

$$u_{\text{C}}(x) \leq -\sigma_c(x) \qquad \forall x \in C \tag{*16}$$

$$u_{\text{D}}(x) \leq 0 \qquad \forall x \in D, \tag{18}$$

and, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$t \geq \gamma_r(t + j) - N_r \quad (t, j) \in \operatorname{dom} \phi. \tag{19}$$

(c) Strict decrease at jumps, nonincreasing during flows: *There exists an LSC function $\sigma_d \in \mathcal{PD}(\mathcal{A})$ such that*

$$u_{\text{C}}(x) \leq 0 \qquad \forall x \in C \tag{20}$$

$$u_{\text{D}}(x) \leq -\sigma_d(x) \qquad \forall x \in D, \tag{*17}$$

and, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$j \geq \gamma_r(t + j) - N_r \quad \forall (t, j) \in \operatorname{dom} \phi. \tag{21}$$

(d) *This item is skipped to keep the enumeration consistent with* [1, Thm. 3.19].

(e) Bounded flow time: *There exist $\lambda \in \mathbb{R}$ and an LSC function $\sigma_d \in \mathcal{PD}(\mathcal{A})$ such that*

$$u_{\text{C}}(x) \leq \lambda V(x) \qquad \forall x \in C \tag{22}$$

$$u_{\text{D}}(x) \leq -\sigma_d(x) \qquad \forall x \in D, \tag{*17}$$

and, for each $r > 0$, there exists $T_r \geq 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$\operatorname{dom} \phi \subset [0, T_r] \times \mathbb{N}. \tag{23}$$

(f) Finite number of jumps: *There exist an LSC function $\sigma_c \in \mathcal{PD}(\mathcal{A})$ and a continuous function $\lambda : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ with $\lambda(0) = 0$ such that*

$$u_{\mathrm{C}}(x) \leq -\sigma_c(x) \qquad \forall x \in C \tag{*16}$$
$$V(g) \leq \lambda(V(x)) \qquad \forall x \in D,\ \forall g \in G(x), \tag{24}$$

and, for each $r \geq 0$, there exists $J_r \in \mathbb{N}$ such that for every solution ϕ to $\mathcal{H}$,

$$\operatorname{dom}\phi \subset \mathbb{R} \times \{0, 1, \ldots, J_r\}. \tag{25}$$

Proof Sketch. The proof proceeds by showing that the assumptions in each case imply the assumptions of the corresponding case in [1, Thm. 3.19(3)]. The class-$\mathcal{K}_\infty$ lower bound on V in [1, Eq. 3.26] holds by assumption, and the upper bound is established by Lemma 2.

Under the assumption that there exists an LSC function $\sigma_c \in \mathcal{PD}(\mathcal{A})$ that satisfies (16), there exists, by Proposition 2, a continuous function $\rho_c \in \mathcal{PD}(0)$ that satisfies [1, Eq. 3.27]. Similarly, if we assume the existence of an LSC function $\sigma_d \in \mathcal{PD}(\mathcal{A})$ that satisfies (17), then there exists a continuous function $\rho_d \in \mathcal{PD}(0)$ that satisfies [1, Eq. 3.28]. □

The next example Theorem 1 can be applied to prove that a set is UGpAS without needing to construct a bound on u_{C} in the form of (1).

Example 3 (Bouncing Ball)**.** Consider a bouncing ball modeled as in [5, Example 3.19] with height $x_1 \geq 0$ and vertical velocity $x_2 \in \mathbb{R}$. The bouncing ball is modeled as the hybrid system $\mathcal{H} := (C, F, D, G)$ with state $x := (x_1, x_2) \in \mathbb{R}^2$ and dynamics given by

$$F(x) := \begin{bmatrix} x_2 \\ -\gamma \end{bmatrix} \qquad \forall x \in C := \{x \in \mathbb{R}^2 \mid x_1 > 0\}$$
$$G(x) := \begin{bmatrix} 0 \\ -\lambda x_2 \end{bmatrix} \quad \forall x \in D := \{x \in \mathbb{R}^2 \mid x_1 = 0, x_2 < 0\},$$

where $\gamma > 0$ is acceleration due to gravity and $\lambda \in [0, 1)$ is the coefficient of restitution when the ball hits the floor. The sets C and D are not closed, so (A1) of the hybrid basic conditions is violated. To show that $\mathcal{A} := \{(0, 0)\}$ is UGpAS, we take the Lyapunov function candidate

$$x \mapsto V(x_1, x_2) := \big(1 + \theta \operatorname{atan}(x_2)\big)\big(x_2^2/2 + \gamma x_1\big)$$

where $\theta := \big(1 - \lambda^2\big)/\big(2 + 2\lambda^2\big)$. Equation (15) holds with

$$s \mapsto \alpha_1(s) := \tfrac{1}{1-\theta} \min\{s^2/4,\ \gamma s/\sqrt{2}\}.$$

Since V is continuously differentiable and F is single valued, we have that for all $x := (x_1, x_2) \in C$,

$$u_{\mathrm{C}}(x) = \langle \nabla V(x),\ F(x)\rangle = -\gamma\theta\left(x_2^2/2 + \gamma x_1\right)/\left(1 + x_2^2\right).$$

Thus, u_{C} is continuous and negative definite with respect to $\mathcal{A}$, and $\sigma_c := -u_{\mathrm{C}}$ satisfies (16). For each $x := (0, x_2) \in D$,

$$u_{\mathrm{D}}(x) = \Big[\lambda^2 - 1 - \theta\big(\operatorname{atan}(\lambda x_2)\lambda^2 + \operatorname{atan}(x_2)\big)\Big]\frac{x_2^2}{2},$$

which is continuous. For any $x \in D \setminus \mathcal{A}$, we have

$$\theta\big(\operatorname{atan}(-\lambda x_2)\lambda^2 - \operatorname{atan}(x_2)\big) \leq \theta(\lambda^2 + 1) < \lambda^2 - 1 < 0,$$

so u_{D} is negative definite. Thus, $\sigma_d := -u_{\mathrm{D}}$ satisfies (17). Therefore, by Theorem 1, $(0, 0)$ is UGpAS for $\mathcal{H}$. ◇

As we saw in Examples 3 and 4, if u_{C} and u_{D} are negative definite and USC, then we can simply use the functions $\sigma_{\mathrm{C}} \equiv -u_{\mathrm{C}}$ and $\sigma_{\mathrm{C}} \equiv -u_{\mathrm{D}}$ for the assumptions in Theorem 1. This approach holds in general if we introduce additional (weak) assumptions on F and G, which are a subset of the hybrid basic conditions. In particular, F is assumed to be locally bounded and OSC, as in (A2), but not assumed to have convex values. The assumptions on G match (A3).

Proposition 3. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$, a compact set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$. Suppose F is OSC and locally bounded, and u_{C} is negative definite with respect to $\mathcal{A}$. Then, $\sigma_c \equiv -u_{\mathrm{C}}$ is LSC and satisfies (16), and there exists a continuous function $\rho \in \mathcal{PD}(0)$ such that $u_{\mathrm{C}}(x) \leq -\rho(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.*

Proposition 4. *Consider a hybrid system $\mathcal{H} = (C, F, D, G)$ on $\mathbb{R}^n$, a compact set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$. Suppose that G is OSC and locally bounded, and u_{D} is negative definite with respect to $\mathcal{A}$. Then, $\sigma_d \equiv -u_{\mathrm{D}}$ is LSC and satisfies (17), and there exists a continuous function $\rho \in \mathcal{PD}(0)$ such that $u_{\mathrm{D}}(x) \leq -\rho(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.*

A. Simplified Assumptions on Hybrid Time Domains

In Theorem 1, the conditions on the hybrid time domain of solutions given in (19) of case (b) and (21) of case (c) are rather non-intuitive and are often difficult to show. Thus, in Propositions 5 and 6, we provide sufficient conditions for (19) and (21), respectively, that are easier to check while remaining general enough to apply to most systems that satisfy (19) or (21).

Proposition 5. *Consider a hybrid system $\mathcal{H}$ and a nonempty closed set $\mathcal{A}$. Suppose that for each $r \geq 0$, there exist $\Delta_T > 0$ and $\Delta_J > 0$ such that for every solution ϕ with $|\phi(0, 0)|_{\mathcal{A}} \in (0, r]$ and for every $(t_0, j_0), (t_1, j_1) \in \operatorname{dom}\phi$,*

$$|t_1 - t_0| \leq \Delta_T \implies |j_1 - j_0| \leq \Delta_J. \tag{26}$$

Then, for each $r \geq 0$, there exist $N_r \geq 0$ and $\gamma_r \in \mathcal{K}_\infty$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \in (0, r]$,

$$t \geq \gamma_r(t + j) - N_r \quad \forall (t, j) \in \operatorname{dom}\phi. \tag{*19}$$

Informally, the assumptions of Proposition 5 state that for every solution that starts within a given distance of $\mathcal{A}$, there exists a bound Δ_J on the number of jumps that can occur during any time interval a given length Δ_T.

Proof Sketch. It can be shown that for each $r \geq 0$ and each solution ϕ to $\mathcal{H}$ with $|\phi(0, 0)|_{\mathcal{A}} \in (0, r]$,

$$j \leq \Delta_J + \frac{\Delta_J}{\Delta_T} t \quad \forall (t, j) \in \operatorname{dom}\phi. \tag{27}$$

Then, by some algebra, we find

$$t \geq \left(\frac{\Delta_T}{\Delta_T+\Delta_J}\right)(t+j) - \frac{\Delta_T\Delta_J}{\Delta_T+\Delta_J} \quad \forall (t,j)\in \operatorname{dom}\phi.$$

Therefore, the conclusion holds with

$$s \mapsto \gamma_r(s) := \left(\frac{\Delta_T}{\Delta_T+\Delta_J}\right)s \quad \text{and} \quad N_r := \frac{\Delta_T\Delta_J}{\Delta_T+\Delta_J}. \qquad \square$$

The analogous result with flows and jumps switched is presented next.

Proposition 6. *Consider a hybrid system $\mathcal{H}$ and a nonempty closed set $\mathcal{A}$. Suppose that for each $r \geq 0$, there exists $\Delta_T > 0$ and $\Delta_J > 0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0,r]$ and for all $(t_0,j_0),(t_1,j_1)\in \operatorname{dom}\phi$,*

$$|j_1 - j_0| \leq \Delta_J \implies |t_1 - t_0| \leq \Delta_T. \tag{28}$$

Then, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0,r]$,

$$j \geq \gamma_r(t+j) - N_r \quad \forall (t,j)\in \operatorname{dom}\phi. \tag{*21}$$

B. Continuous-Time and Discrete-Time Systems

The following corollaries are special cases of Theorem 1 for continuous- and discrete-time systems.

Corollary 1 (Continuous-time Lyapunov Theorem)**.** *Consider compact set $\mathcal{A}\subset\mathbb{R}^n$, a continuous-time system on $C\subset\mathbb{R}^n$*

$$\dot{x} \in F(x) \quad x \in C, \tag{29}$$

and a Lyapunov function candidate V with respect to $\mathcal{A}$ for (29). Suppose there exists $\alpha_1 \in \mathcal{K}_\infty$ and an LSC function $\sigma_c \in \mathcal{PD}(\mathcal{A})$ such that

$$\alpha_1(|x|_{\mathcal{A}}) \leq V(x) \quad \text{and} \quad u_{\mathrm{C}}(x) \leq -\sigma_c(x) \quad \forall x \in C.$$

Then, $\mathcal{A}$ is UGpAS for (29).

Corollary 2 (Discrete-time Lyapunov Theorem)**.** *Consider compact set $\mathcal{A}\subset\mathbb{R}^n$, a discrete-time system on $D\subset\mathbb{R}^n$*

$$x^+ \in G(x) \quad x \in D, \tag{30}$$

and Lyapunov function candidate V with respect to $\mathcal{A}$ for (30). Suppose there exists $\alpha_1 \in \mathcal{K}_\infty$ and an LSC function $\sigma_d \in \mathcal{PD}(\mathcal{A})$ such that

$$\begin{aligned} \alpha_1(|x|_{\mathcal{A}}) &\leq V(x) && \forall x \in D \cup G(D) \\ u_{\mathrm{D}}(x) &\leq -\sigma_d(x) && \forall x \in D. \end{aligned}$$

Then, $\mathcal{A}$ is UGpAS for (30).

The next example illustrates how Theorem 1 can be used to show that a compact set is UGpAS for $\dot{x} = F(x)$ with F discontinuous.

Example 4. Consider the continuous-time system

$$\dot{x} = F(x) := -\lfloor x \rfloor \quad x \in C := \mathbb{R},$$

where $\lfloor x \rfloor$ is the largest integer m such that $m \leq x$ and $\lceil x \rceil$ is the smallest integer n such that $n \geq x$. The $\lfloor\cdot\rfloor$ function is USC and $\lceil\cdot\rceil$ is LSC. Let $\mathcal{A} := [0,1]$ and consider $x \mapsto V(x) := |x|^2_{\mathcal{A}}$. We find that

$$u_{\mathrm{C}}(x) = \begin{cases} (-|x|_{\mathcal{A}})\lfloor x \rfloor & \text{if } x \geq 0 \\ (|x|_{\mathcal{A}})\lfloor x \rfloor & \text{if } x < 0, \end{cases}$$

which is neither LSC nor USC. Let

$$-\sigma_c(x) := \begin{cases} |x|_{\mathcal{A}}(1-\lceil x \rceil) & \text{if } x \geq 0 \\ (|x|_{\mathcal{A}})\lfloor x \rfloor & \text{if } x < 0. \end{cases}$$

We see $-\sigma_c$ is USC, so σ_c is LSC. For $x \leq 0$, $u_{\mathrm{C}}(x) = -\sigma_c(x)$, and for $x \geq 0$, $-\lfloor x \rfloor \geq 1 - \lceil x \rceil$, so $u_{\mathrm{C}}(x) \leq -\sigma_c(x)$, thus (16) holds. It can be easily checked that $\sigma_c \in \mathcal{PD}(\mathcal{A})$. Therefore, $\mathcal{A}$ is UGpAS for $\dot{x} = F(x)$, by Corollary 1. ◇

IV. Conclusion

In this paper, we presented a relaxation of the hybrid Lyapunov theorem, along with corollaries for continuous-time and discrete-time systems. One avenue for future work is to find an alternative function definition to u_{C} in (13). In particular, we have found systems with where u_{C} is positive at points where V cannot increase, indicating u_{C} is an over-conservative upper bound.

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