Exploiting Uncertified Controllers via Uniting Feedback

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Introduction – Cyber-Physical Systems

Cyber-physical systems are electromechanical systems that include digital electronics (e.g., sensors and computers) that interact with physical components or processes.

Heating and Air Conditioning

Spacecraft

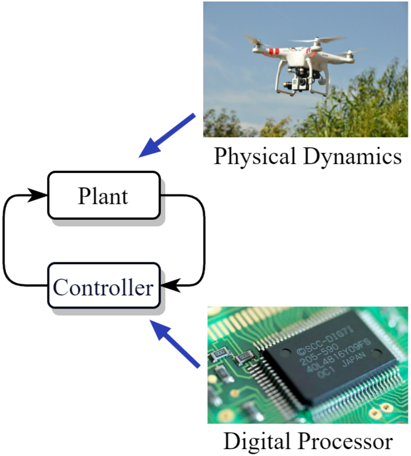
Walking Robots

Chemical Plants

Quadcopters

Autonomous Cars

Introduction – Cyber-Physical Systems Control Problems



Fundamental Problems of Control Theory

- Don't Crash
- Get Where You're Going
- Minimize cost/Maximize rewards

Complicating Factors

- Complex dynamics
- Uncertainty
- Dynamic environments
- Computational limitations
- ...

Introduction – Spectrum of Control Methods

Experiments

Reinforcement Learning

Simulations

Statistical Verification

Conformal prediction

Model Predictive Control

Barrier Functions

Lyapunov Functions

Reachability Analysis

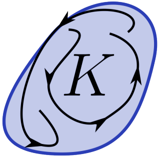
← Easier to apply to complex systems

Stronger verification guarantees →

⟹ How can we combine these approaches?

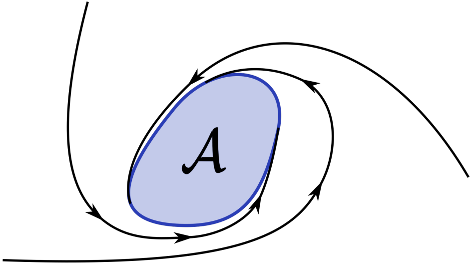
Control Objectives

Forward Invariance
("Don't Crash")



Every trajectory remains in $\mathcal{A}$.

Global Asymptotic Stability
("Get Where You're Going")

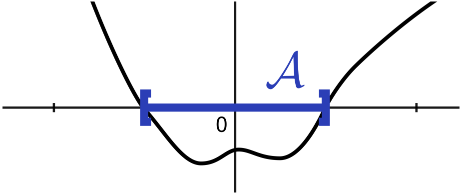


Every trajectory moves toward $\mathcal{A}$.

Certificate Functions

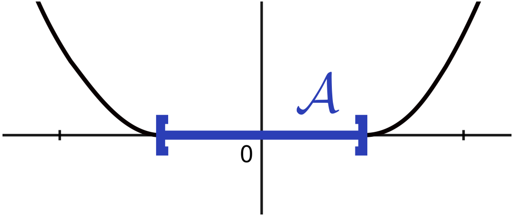
To prove forward invariance and asymptotic stability, we use *certificate functions*.

Forward Invariance



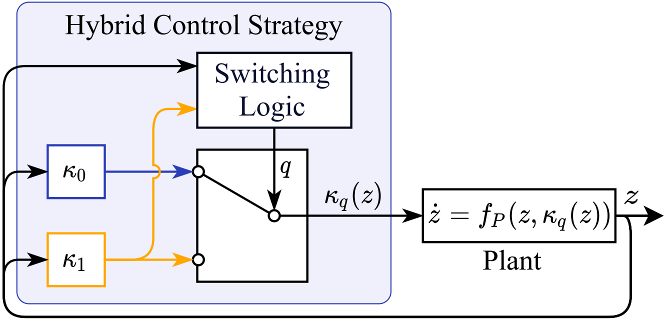
Certificate: Barrier Function

Asymptotic Stability



Certificate: Lyapunov Function

Uniting Feedback Control



κ_0 : certified controller, guaranteed to achieve control objective

κ_1 : any controller

Why Use an Uncertified Controller?

Certificates are hard to construct.

We may have an “advanced” controller that is difficult/impossible to certify but is

- Uses less energy
- Produces faster convergence
- Requires less computation
- Explores a region, collecting measurements.

Examples:

- Linear quadratic regulator (LQR) for the linearization of a system.
- Model predictive control (MPC) with computational delays.
- Black box controllers (e.g., neural network controllers).

Introduction – Previous Uses of Switched Controllers

Sometimes, a single continuous controller cannot satisfy design requirements.

Switching has been used to...

- Achieve robust global asymptotic stability around obstructions.[1]
- Unite multiple Lyapunov-certified controllers (such as local and global controllers) to achieve global asymptotic stability.[2]

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[1] Mayhew, Ricardo G. Sanfelice, and Teel (2011), "Quaternion-Based Hybrid Control for Robust Global Attitude Tracking".
Ricardo G. Sanfelice et al. (2006), "Robust Hybrid Controllers for Continuous-Time Systems with Applications to Obstacle Avoidance and Regulation to Disconnected Set of Points".

[2] Prieur (2001), "Uniting Local and Global Controllers with Robustness to Vanishing Noise".
Teel and Kapoor (1997), "Uniting Local and Global Controllers".
El-Farra, Mhaskar, and Christofides (2005), "Output Feedback Control of Switched Nonlinear Systems Using Multiple Lyapunov Functions".

Simplex architecture

The *Simplex architecture* is an approach for switching between an “advanced,” unverified controller and a “simple,” easy-to-verify controller.[3]

Barrier functions have been used with the Simplex architecture to guarantee safety for hybrid systems while using an unverified controller.

Prior approaches have drawbacks:

- Requires costly reachability analysis and only defines “one way” switching.[4]
- Only rectangular constraints are considered, and the switching criteria depends on the extremal values of the vector field over the entire admissible set.[5]

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[3] Rivera et al. (1996), *An Architectural Description of the Simplex Architecture.*
Seto et al. (1998), “The Simplex Architecture for Safe Online Control System Upgrades”.

[4] Yang et al. (2017), “A Simplex Architecture for Hybrid Systems Using Barrier Certificates”.

[5] Damare et al. (2022), “A Barrier Certificate-Based Simplex Architecture with Application to Microgrids”.

Outline

Outline

Uniting Feedback for Safety

Uniting Feedback for Global Asymptotic Stability

Relaxed Lyapunov Conditions

Uniting Feedback with Hybrid Controllers and Hybrid Plants

Software Tools

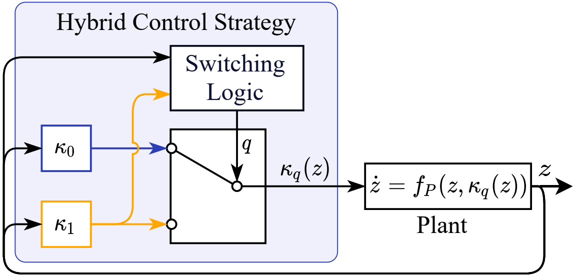
Conclusion

Uniting Feedback for Forward Invariance

Given: A nonlinear plant

$$\dot{z} = f_{\mathrm{P}}(z, u), \quad z \in \mathbb{R}^{n_{\mathrm{P}}}, \; u \in \mathbb{R}^{m_{\mathrm{P}}}.$$

and controllers
κ_0: barrier-certified to render K forward invariant
κ_1: any controller

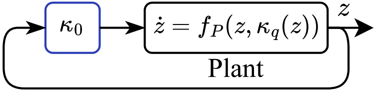


Goal: Design switching logic for $q \in \{0, 1\}$ such that

- K is forward invariant.
- κ_1 is preferred over κ_0.
- Switching does not chatter.

Barrier Function Certificate

$$\dot{z} = f_0(z) := f_{\mathrm{P}}(z, \kappa_0(z)).$$



Has a *barrier function* B that certifies

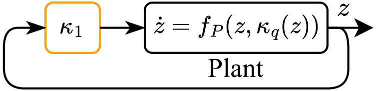
$$K = \{z \in \mathbb{R}^n \mid B(z) \leq 0\}$$

is forward invariant.
On some neighborhood U of K,

$$\dot{B}_0(z) := \langle \nabla B(z), f_0(z) \rangle \leq 0$$

for all $z \in U \setminus K$.

$$\dot{z} = f_1(z) := f_{\mathrm{P}}(z, \kappa_1(z)).$$



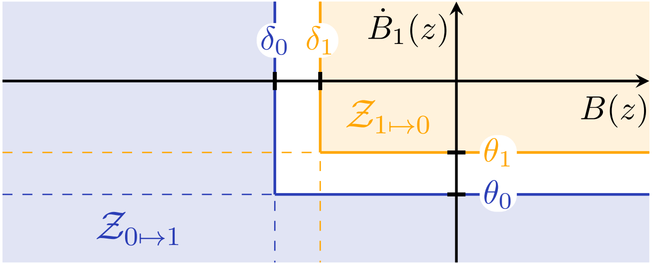
Rate of change of B when using κ_1:

$$\dot{B}_1(z) := \langle \nabla B(z), f_1(z) \rangle = ??.$$

Switching Criteria and Hold Criteria

Pick four *thresholds* δ_0, δ_1, θ_0, θ_1, such that

$$\delta_0 < \delta_1 \leq 0 \quad \text{and} \quad \theta_0 < \theta_1 \leq 0.$$



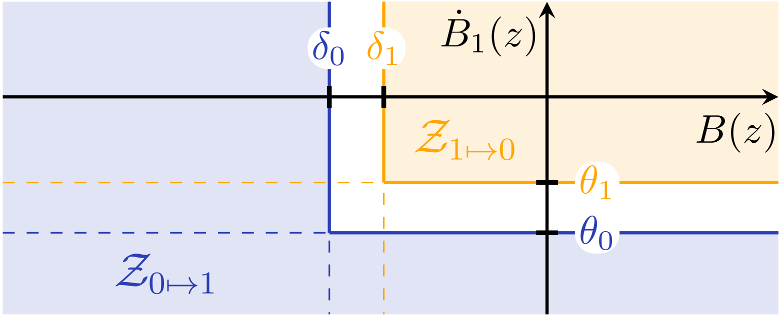
For $q = 0$ (certified controller):

- Hold $q = 0$ if $z \in \mathcal{Z}_0$.
- Switch to $q = 1$ if $z \in \mathcal{Z}_{0\mapsto 1}$.

For $q = 1$ (uncertified controller):

- Hold $q = 1$ if $z \in \mathcal{Z}_1$.
- Switch to $q = 0$ if $z \in \mathcal{Z}_{1\mapsto 0}$.

Switching Criteria and Hold Criteria



$$\mathcal{Z}_{1\mapsto 0} := \{z \in \mathbb{R}^n \mid B(z) \geq \delta_1, \ \dot{B}_1(z) \geq \theta_1\}$$

$$\mathcal{Z}_{0\mapsto 1} := \{z \in \mathbb{R}^n \mid B(z) \leq \delta_0 \text{ or } \dot{B}_1(z) \leq \theta_0\}.$$

Dynamics of Closed-Loop System

Between switches:

- z evolves according to $\dot{z} = f_{\mathrm{P}}(z,\ \kappa_q\ (z))$
- q is constant

At each switch:

- z is unchanged
- q is toggled to the opposite value in $\{0, 1\}$

Example: Linear and Affine Feedbacks

Consider the double integrator

$$\dot{z} = f_{\mathrm{P}}(z, u) := \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} z + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u.$$

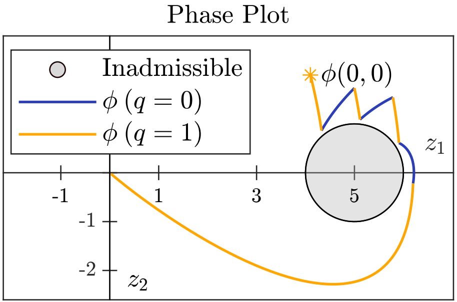
Admissible set:

$$K := \left\{ z \in \mathbb{R}^2 : |z - (5, 0)| \geq 1 \right\}.$$

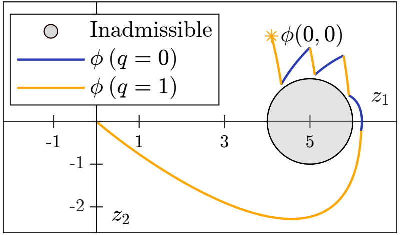
Controllers:

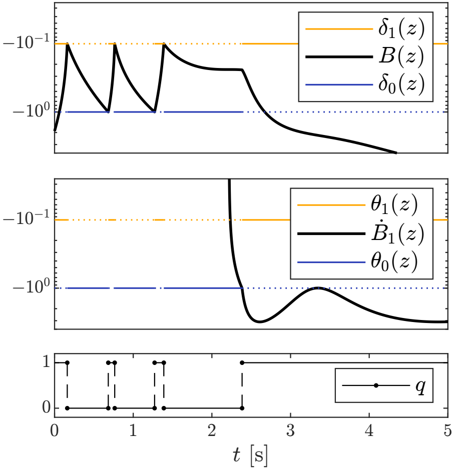
$$\kappa_0(z) = \begin{bmatrix} -1 & 1 \end{bmatrix} (z - c) \quad \text{(certified)}$$

$$\kappa_1(z) = \begin{bmatrix} -1 & -2 \end{bmatrix} z \quad \text{(uncertified)}$$

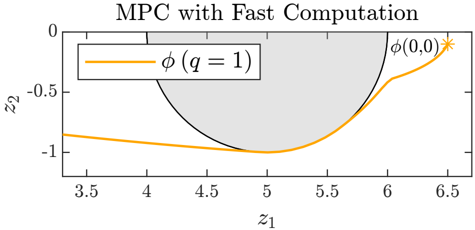


Example: Linear and Affine Feedbacks

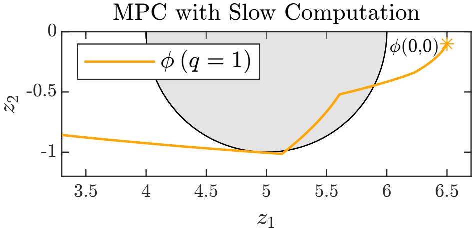

Phase Plot


Switching Criteria

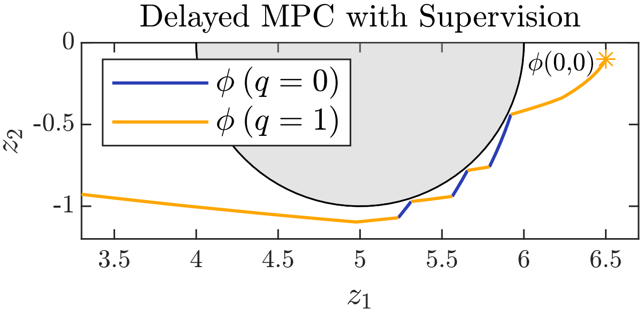
Example: MPC with Computational Delays



MPC with Fast Computation



MPC with Slow Computation



Delayed MPC with Supervision

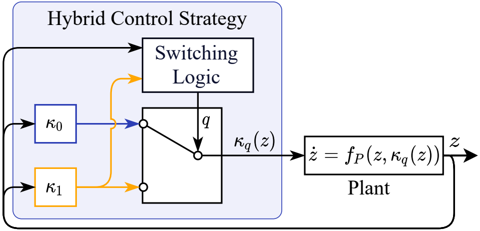
Theorem: Forward Invariance

Suppose that

- B is a continuously differentiable barrier function of K for $\dot{z} = f_0(z)$.
- f_0 and f_1 are continuous.

Then, K is forward invariant w.r.t. z for the closed-loop system.

Closed-loop System



Remark. We also give conditions to ensure that

- Solutions exist for all $t \geq 0$.
- The time between switches is not too short.

Outline

Problem Setting

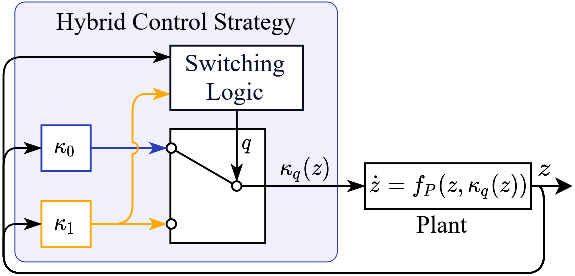
Given: A control system

$$\dot{z} = f_{\mathrm{P}}(z, u), \quad z \in \mathbb{R}^{n_{\mathrm{P}}}, u \in \mathbb{R}^{m_{\mathrm{P}}}.$$

and controllers

κ_0: Lyapunov-certified to render $\mathcal{A}$ globally asymptotically stable

κ_1: any controller



Goal: Design switching logic for $q \in \{0, 1\}$ such that

- $\mathcal{A}$ is globally asymptotically stable
- κ_1 is preferred over κ_0
- switching does not chatter.

Problem Setting – Lyapunov-certified Controller

For the *Lyapunov-certified* controller κ_0, there exists a Lyapunov function

$$V_{\mathrm{P}} : \mathbb{R}^{n_{\mathrm{P}}} \to [0, \infty)$$

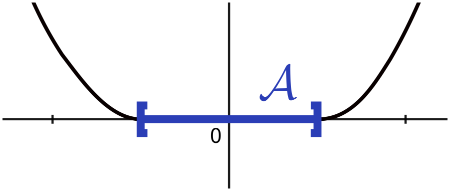
for

$$\dot{z} = f_0(z) := f_{\mathrm{P}}(z, \kappa_0(z)),$$

that guarantees $\mathcal{A}$ is globally asymptotically stable.

Value of V_{P} decreases outside $\mathcal{A}$:

$$\dot{V}_0 := \langle \nabla V_{\mathrm{P}}, f_0(z) \rangle \leq -\sigma_0(z).$$



For the *uncertified* controller κ_1, no assumptions on the rate of change of V_{P},

$$\dot{V}_1 := \langle \nabla V_{\mathrm{P}}, f_1(z) \rangle = ??$$

for

$$\dot{z} = f_1(z) := f_{\mathrm{P}}(z, \kappa_1(z)).$$

Example: Model Predictive Controller with Slow Computation

Consider a nonlinear plant

$$\dot{z} = f_{\mathrm{P}}(z, u)$$

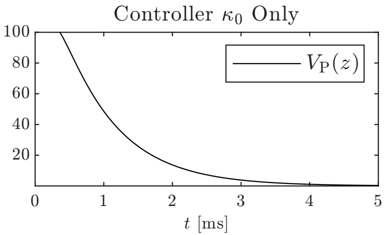
and two controllers:

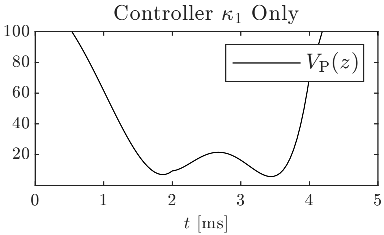
κ_0: Lyapunov-certified controller

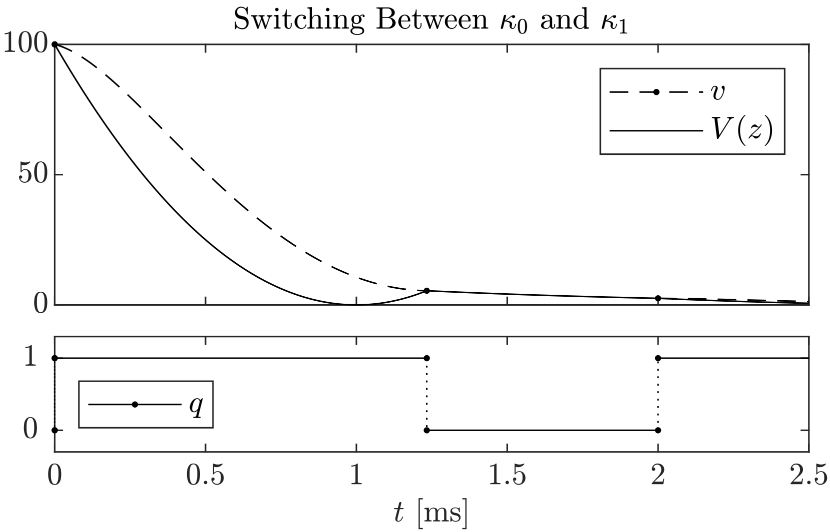
κ_1: Model predictive controller (MPC) with a sampling period of 1 ms

Suppose new MPC feedback value is not available at 1 ms.

When should we switch?


Controller κ_0 Only


Controller κ_1 Only



- The dynamics of v are described later.

Hybrid Control Strategy – Switching Logic

Buffer function: Pick a continuous, positive function $x \mapsto \delta(x) > 0$

For $q = 0$ (certified controller):

V_{P} is “small enough to switch to $q = 1$” if

$$V_{\mathrm{P}}(z) + \delta(z) \leq v$$

V_{P} is “large enough to hold $q = 0$” if

$$V_{\mathrm{P}}(z) + \delta(z) \geq v$$

For $q = 1$ (uncertified controller):

V_{P} is “small enough to hold $q = 1$” if

$$V_{\mathrm{P}}(z) \leq v$$

V_{P} is “large enough to switch to $q = 0$” if

$$V_{\mathrm{P}}(z) \geq v$$

Example: Switching Logic

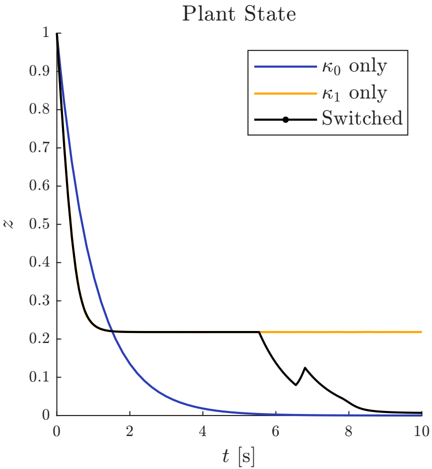
Consider the plant

$$\dot{z} = u$$

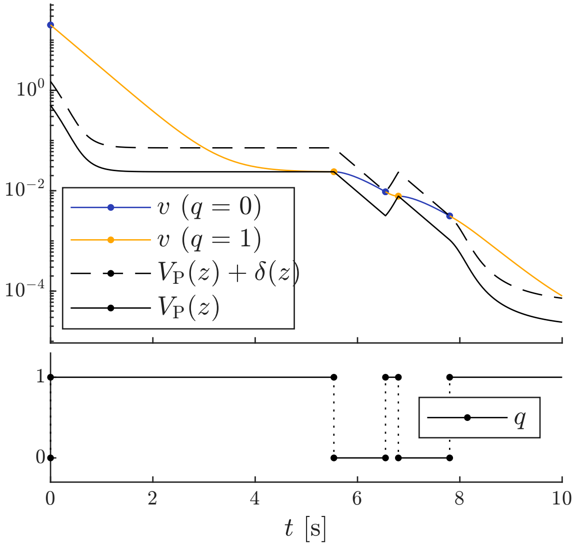
with $z > 0$ and $u \in \mathbb{R}$ and controllers

$$\kappa_0(z) := -z$$

$$\kappa_1(z) := -2\sin\left(\frac{1}{z + 0.1}\right)$$



Switching Criteria



Dynamics of Auxiliary v Variable

At each switch:

- v is set to $\max\{V_{\mathrm{P}}(z), v\}$

Between switches:

- v evolves according to

$$\dot{v} := -\gamma \tanh(v)\sigma_0(z) - \mu\big(v - V_{\mathrm{P}}(z)\big),$$

where $\gamma > 0$, $\mu > 0$, and σ_0 is continuous and positive definite.

- v converges to 0.
- If $q = 0$ and $v < V_{\mathrm{P}}(z)$, then v can increase gradually.

Example: Linear Quadratic Regulator of Linearized System

Consider the nonlinear plant

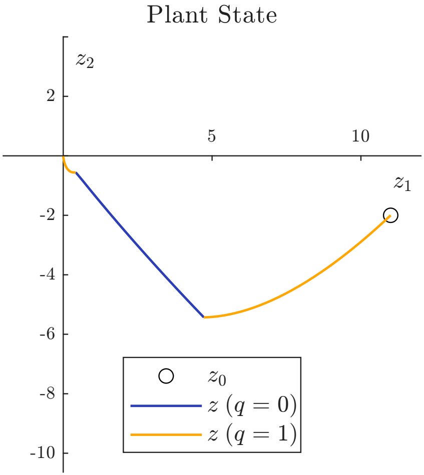
$$\dot{z} = Az + Bu + \underbrace{f(z,u)}_{\text{Nonlinear component}} .$$

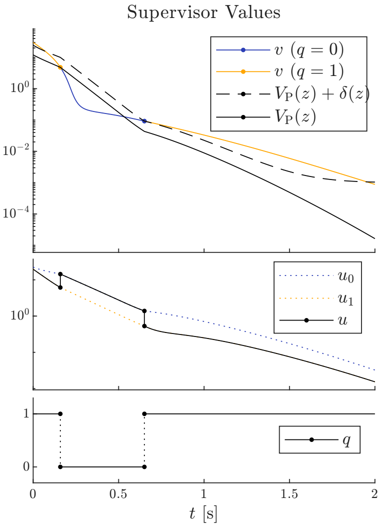
Let κ_0 be an (inefficient) Lyapunov-certified controller.

Let κ_1 be the LQR feedback that solves the following LQR problem:

$$\begin{aligned} \underset{u(\cdot)}{\text{minimize}} \quad & \int_0^\infty |z(t)|^2 + |u(t)|^2 \, dt \\ \text{subject to} \quad & \dot{z} = Az + Bu. \end{aligned}$$

Example: LQR of Linearized System





Theorem:[6] Global Asymptotic Stability

Suppose that

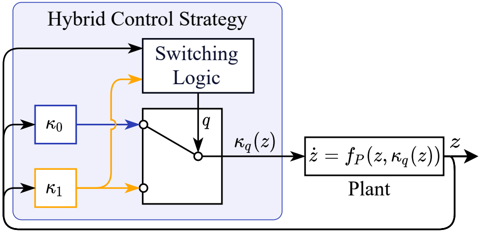
- $\mathcal{A}$ is compact;
- f_0 and f_1 are continuous;
- V_{P} is a Lyapunov function for $\dot{z} = f_0(z)$.

Then,

$$\widetilde{\mathcal{A}} := \{(z, v, q) \mid z \in \mathcal{A}, v = 0\}$$

is (uniformly) globally asymptotically stable for the closed-loop system.

Closed-loop System



Remark. The asymptotic stability of $\widetilde{\mathcal{A}}$ is robust to small perturbations.

[6] Paul K. Wintz, Ricardo G. Sanfelice, and Hespanha (2022), "Global Asymptotic Stability of Nonlinear Systems While Exploiting Properties of Uncertified Feedback Controllers via Opportunistic Switching".

Summary

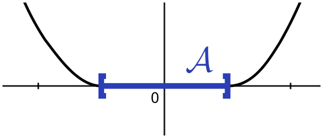
- Lyapunov-certified controller acts as a backup to ensure convergence while using an uncertified controller.
- Exploit useful properties of **any** uncertified continuous controller without losing the convergence guarantee.

Next steps

- We consider more general systems later.

Outline

Lyapunov Theorems



Given: A dynamical system and a set $\mathcal{A} \in \mathbb{R}^n$.

Goal: Prove that $\mathcal{A}$ is asymptotically stable.

Method: Construct $V : \mathbb{R}^n \to [0, \infty)$ such that

1. V is positive definite with respect to $\mathcal{A}$.
2. $t \mapsto V(x(t))$ is decreasing for each solution $t \mapsto x(t)$ while $x(t) \notin \mathcal{A}$.

If f is Lipschitz continuous, then the "decreasing" condition for $\dot{x} = f(x)$ is

$$\dot{V}(x) := \langle \nabla V(x),\, f(x) \rangle < 0.$$

Lyapunov-like Theorems (Non-smooth Systems)

We also consider *non-smooth* systems:

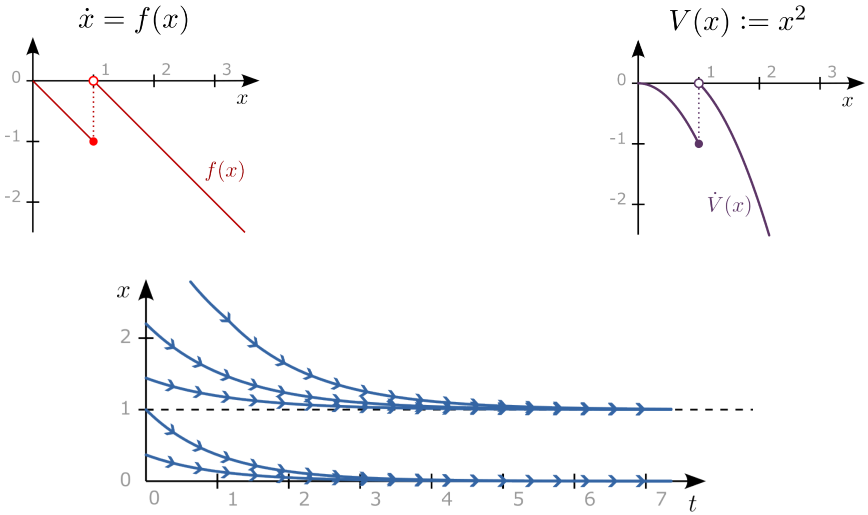
1. $\dot{x} = f(x)$ with f discontinuous (Non-smooth ODE)
2. $\dot{x} \in F(x)$ (differential inclusion)
3. $x^+ \in G(x)$ (difference inclusion)
4. $\mathcal{H} : \begin{cases} \dot{x} \in F(x) & \forall x \in C \\ x^+ \in G(x) & \forall x \in D. \end{cases}$ (hybrid system)

We can also have V non-differentiable.

For non-smooth systems,

$$\left(\dot{V}(x) < 0 \text{ for all } x \notin \mathcal{A}\right) \;\not\Longrightarrow\; \left(\mathcal{A} \text{ is globally asymptotically stable}\right)$$

Example: $\dot{V} < 0$ without convergence to 0

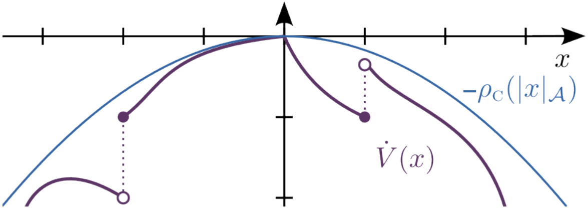


Lyapunov-like Theorems (Nonsmooth Systems)

For nonsmooth systems, prior results replace "$\dot{V}(x) < 0$" with

$$\dot{V}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \quad \forall x \in \mathbb{R}^n, \tag{1}$$

where $\rho_{\mathrm{C}} : [0, \infty) \to [0, \infty)$ is continuous and positive definite.



Lyapunov-like theorems – Relaxed Lyapunov Condition

It is often difficult to construct ρ_{C}.

- Must be continuous.
- Must be a function of the *distance* from $\mathcal{A}$.

When $\mathcal{A}$ is compact, we found a relaxation of the Lyapunov conditions.[7]

If there exists $\sigma_{\mathrm{LSC}} : \mathbb{R}^n \to [0, \infty)$ such that

- σ_{LSC} is a function of x (instead of $|x|_{\mathcal{A}}$),
- σ_{LSC} is lower semicontinuous (instead of continuous),
- σ_{LSC} is positive definite with respect to $\mathcal{A}$, and
- $\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x)$...

...then there exists ρ_{C} (continuous and positive definite) such that

$$\dot{V}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}}).$$

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[7] Paul K Wintz and Ricardo G Sanfelice (2025), "Relaxed Lyapunov Conditions".

Problem Statement: Construction of ρ_{C}

Given: $\mathcal{A}$ (compact) and σ_{LSC}

Goal: Construct ρ_{C} such that

$$\sigma_{\mathrm{LSC}}(x) \geq \quad \rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n.$$

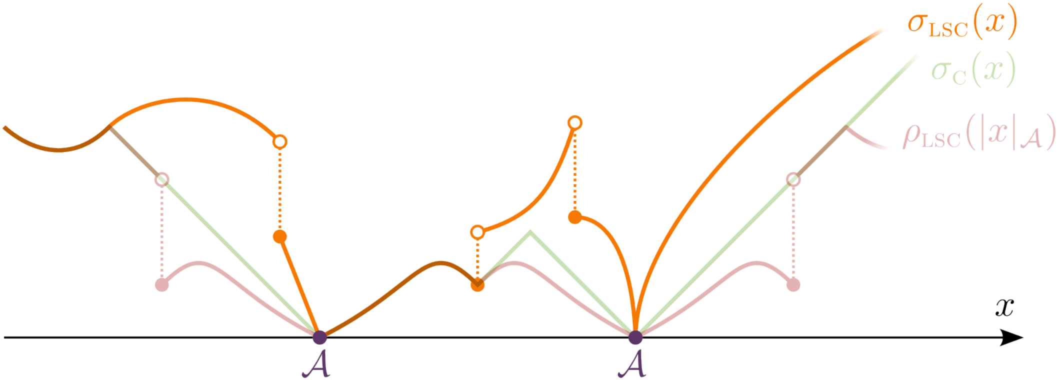
$$-\sigma_{\mathrm{LSC}}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \qquad \forall x \in \mathbb{R}^n.$$

Then,

$$\left(\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x)\right) \implies \left(\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x) \leq -\rho_{\mathrm{C}}(|x|_{\mathcal{A}})\right).$$

Construction of ρ_{C} Outline

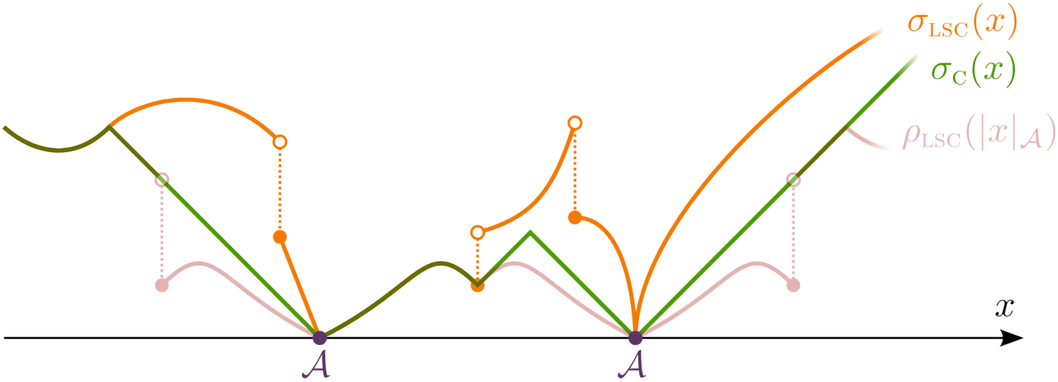
$\sigma_{\mathrm{LSC}}(x)$ $> 0 \qquad \forall x \notin \mathcal{A}.$



Construction of ρ_{C} Outline

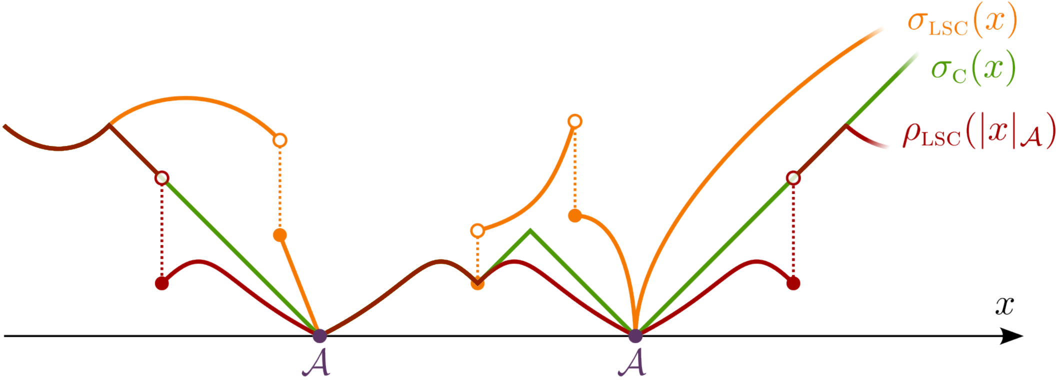
$\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x) \qquad > 0 \qquad \forall x \notin \mathcal{A}.$

continuous



Construction of ρ_{C} Outline

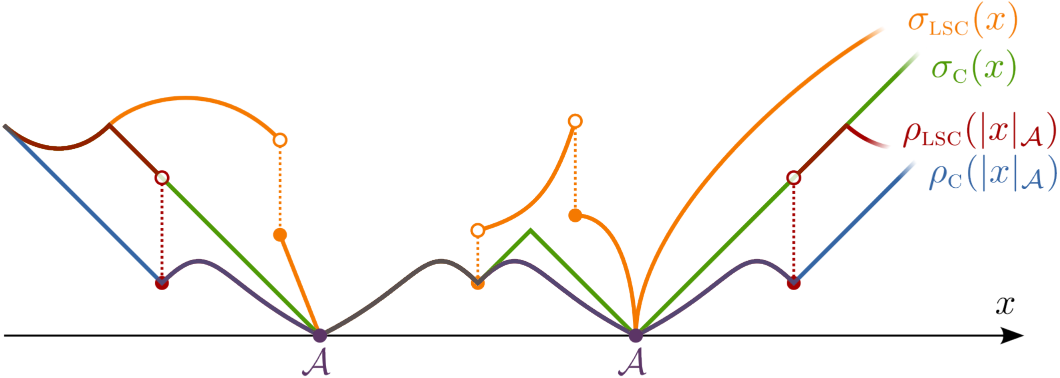
$$\sigma_{\mathrm{LSC}}(x) \geq \underbrace{\sigma_{\mathrm{C}}(x)}_{\text{continuous}} \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \qquad\qquad > 0 \qquad \forall x \notin \mathcal{A}.$$



Construction of ρ_{C} Outline

$$\underbrace{\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)}_{} \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}}) > 0 \qquad \forall x \notin \mathcal{A}.$$

continuous (under $\sigma_{\mathrm{C}}(x)$) — continuous (under $\rho_{\mathrm{C}}(|x|_{\mathcal{A}})$)



Given $\sigma_{\text{LSC}}(x)$.

Make continuous

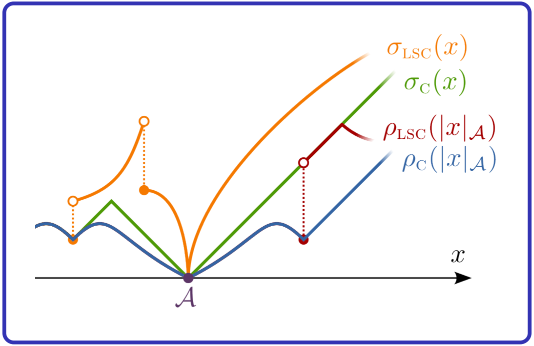
$$\sigma_{\text{C}}(x) := \min_{x' \in \mathbb{R}} \big(\sigma_{\text{LSC}}(x') + |x' - x| \big).$$

Make function of distance (but LSC)

$$\rho_{\text{LSC}}(r) := \min_{x \in \mathbb{R}^n} \big\{ \sigma_{\text{C}}(x) : |x|_{\mathcal{A}} = r \big\}.$$

Make continuous, function of distance

$$\rho_{\text{C}}(r) := \min_{r' \geq 0} \big(\rho_{\text{LSC}}(r) + |r' - r| \big).$$



Relaxed Lyapunov Theorem for Continuous-time Systems

Consider a hybrid system $\dot{x} \in F(x)$ on $\mathbb{R}^n$, a nonempty compact set $\mathcal{A} \subset \mathbb{R}^n$, and a Lyapunov function candidate V with respect to $\mathcal{A}$ for $\mathcal{H}$.

Suppose that

1. there exists $\alpha \in \mathcal{K}_\infty$ such that $\alpha(|x|_\mathcal{A}) \leq V(x)$ for all $x \in \mathbb{R}^n$, and
2. there exist LSC function σ_{LSC} that is positive definite w.r.t. $\mathcal{A}$ such that

$$\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x) \qquad\qquad \forall x \in \mathbb{R}^n.$$

Then, $\mathcal{A}$ is (uniformly) globally asymptotically stable for $\dot{x} \in F(x)$.

How pick a good choice for σ_{LSC}

For any function $f : \mathbb{R}^n \to \mathbb{R}$, the function

$$x \mapsto \liminf_{x' \to x} f(x')$$

is lower semicontinuous and

$$\liminf_{x' \to x} f(x') \leq f(x) \qquad \forall x \in \operatorname{dom} f$$

Thus, if we pick

$$x_0 \mapsto \sigma_{\mathrm{LSC}}(x_0) := \liminf_{x \to x_0} \Big(-\dot{V}(x) \Big),$$

then

$$\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x_0).$$

Example (Continuous-time with discontinuous f)

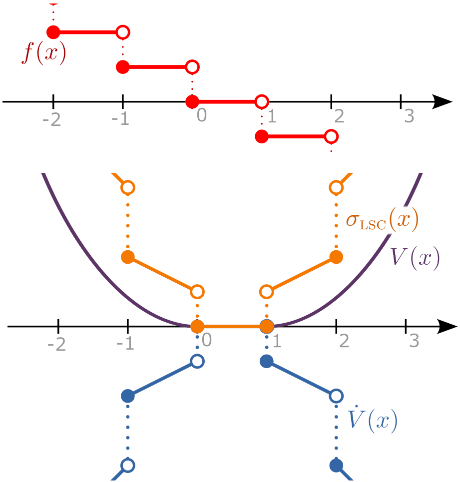
Consider the continuous-time system

$$\dot{x} = f(x) := -\lfloor x \rfloor \quad \forall x \in \mathbb{R},$$

Let $\mathcal{A} := [0, 1]$ and $x \mapsto V(x) := |x|^2_{\mathcal{A}}$.
Let

$$\sigma_{\mathrm{LSC}}(x) := \liminf_{x' \to x} -\dot{V}(x)$$

$\implies$ $\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x)$.
$\implies$ $\mathcal{A}$ is globally asymptotically stable.



Summary

We presented

- relaxation of Lyapunov conditions
- several insertion theorems for positive definite functions.

Future Work

- Generalize relaxed Lyapunov conditions:
    - consider $\mathcal{A}$ non-compact
    - Other types of Lyapunov functions, e.g., ISS Lyapunov functions.

Outline

Uniting Feedback for Safety

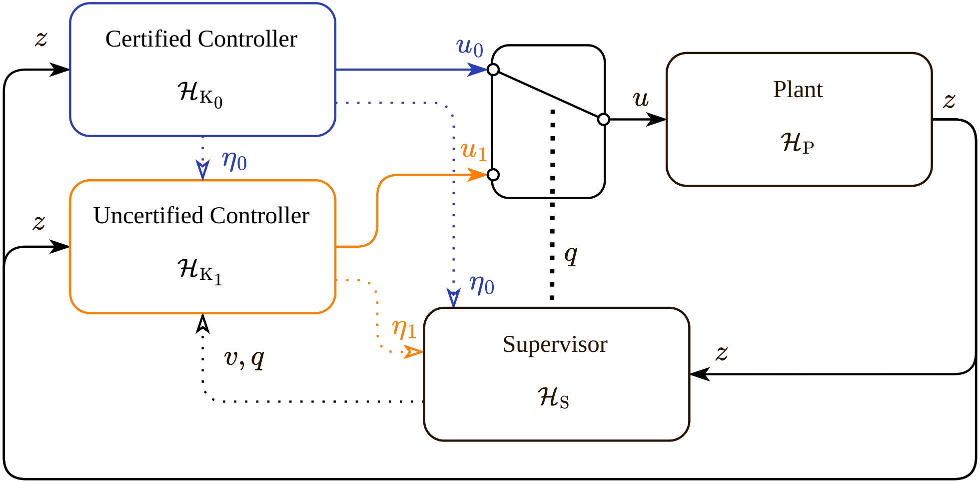
Uniting Feedback for Global Asymptotic Stability

Relaxed Lyapunov Conditions

Uniting Feedback with Hybrid Controllers and Hybrid Plants

Software Tools

Conclusion



Certified Controller

$\mathcal{H}_{\mathrm{K}_0}$

Uncertified Controller

$\mathcal{H}_{\mathrm{K}_1}$

Plant

$\mathcal{H}_{\mathrm{P}}$

Supervisor

$\mathcal{H}_{\mathrm{S}}$

Hybrid Systems as Models of Cyber-Physical Systems

Hybrid dynamical systems are a type of mathematical model of dynamical systems that combine continuous flows and discrete jumps.

Hybrid dynamical systems are a natural choice for modeling cyber-physical systems.

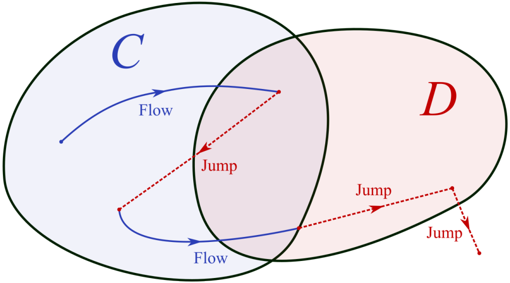
flows	▶ continuous evolution of physical state.
jumps	▶ evolution of digital components ▶ impacts in physical states

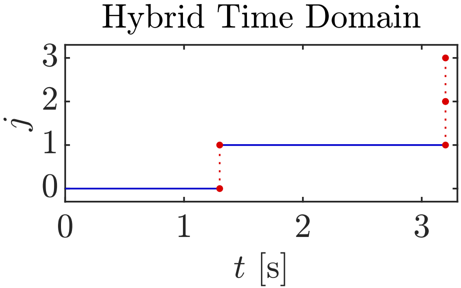
Hybrid Dynamical Systems

$$\mathcal{H}: \begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D \end{cases}$$

- flow set $C \subset \mathbb{R}^n$
- flow map $f : C \to \mathbb{R}^n$

- jump set $D \subset \mathbb{R}^n$
- jump map $g : D \to \mathbb{R}^n$





Hybrid Dynamical Systems with Set-valued Dynamics

$$\mathcal{H} : \begin{cases} \dot{x} \in F(x) & x \in C \\ x^{+} \in G(x) & x \in D \end{cases}$$

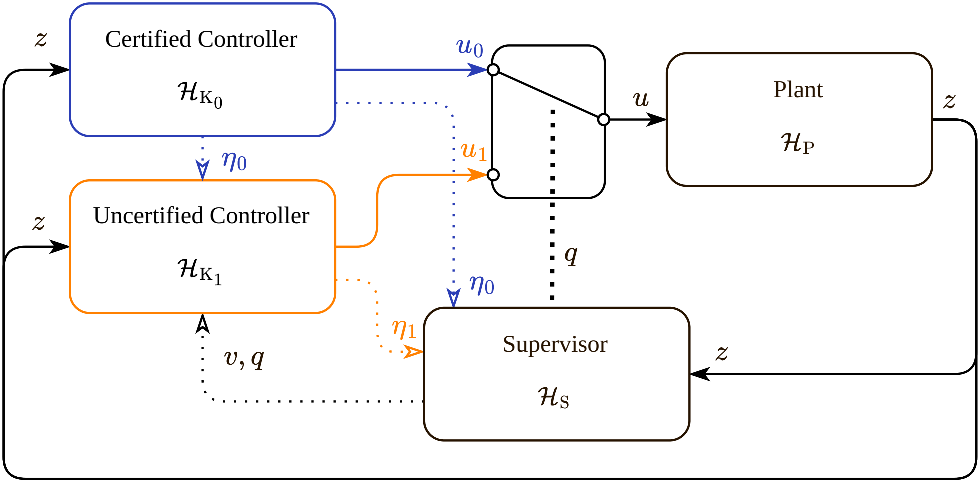
Hybrid Solutions

A *solution* ϕ to $\mathcal{H}$ is defined on a hybrid time domain $\operatorname{dom}\phi \subset [0, \infty) \times \mathbb{N}$:

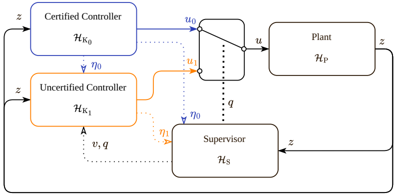
$$\operatorname{dom}\phi = ([t_0, t_1] \times \{0\}) \cup ([t_1, t_2] \times \{1\}) \cup \cdots .$$
$$0 = t_0 \leq t_1 \leq t_2 \leq \cdots .$$

A hybrid arc $(t, j) \mapsto \phi(t, j)$ is a solution to $\mathcal{H}$ if

- $\dot{\phi}(t, j) \in F(\phi(t, j))$ for almost all t in each intervals of flow $[t_j, t_{j+1}]$
- $\phi(t_j, j + 1) \in G(\phi(t_j, j))$ for each jump time t_j in $\operatorname{dom}(\phi)$.



Certified Controller
$\mathcal{H}_{\mathrm{K}_0}$
Uncertified Controller
$\mathcal{H}_{\mathrm{K}_1}$
Plant
$\mathcal{H}_{\mathrm{P}}$
Supervisor
$\mathcal{H}_{\mathrm{S}}$
z
z
z
z
u_0
u_1
u
η_0
η_0
η_1
q
v, q



Plant

$$\mathcal{H}_{\mathrm{P}} : \begin{cases} \dot{z} \in F_{\mathrm{P}}(z, u) & (z, u) \in C_{\mathrm{P}} \\ z^{+} \in G_{\mathrm{P}}(z, u) & (z, u) \in D_{\mathrm{P}} \end{cases}$$

Certified Controller

$$\mathcal{H}_{\mathrm{K}_0} : \begin{cases} \dot{\eta}_0 \in F_{\mathrm{K}_0}(z, \eta_0) & (z, \eta_0) \in C_{\mathrm{K}_0} \\ \eta_0^{+} \in G_{\mathrm{K}_0}(z, \eta_0) & (z, \eta_0) \in D_{\mathrm{K}_0} \\ u_0 = \kappa_0(z, \eta_0). \end{cases}$$

Uncertified Controller

$$\mathcal{H}_{\mathrm{K}_1} : \begin{cases} \dot{\eta}_1 \in F_{\mathrm{K}_1}(x) & x \in C_{\mathrm{K}_1} \\ \eta_1^{+} \in G_{\mathrm{K}_1}(x) & x \in D_{\mathrm{K}_1} \\ u_1 \in \kappa_1(x), \end{cases}$$

Supervisor Design for Global Asymptotic Stability

Given a Lyapunov function $(z, \eta_0) \mapsto V_{\mathrm{P}}(z, \eta)$ for $\mathcal{H}_{\mathrm{P}} \times \mathcal{H}_{\mathrm{K}_0}$ that certifies that $\mathcal{A}_{\mathrm{P}}$ is asymptotically stable for (z, η_0).

We extended the previous supervisor design as follows:

$$\mathcal{H}_{\mathrm{S}} : \begin{cases} \begin{bmatrix} \dot{v} \\ \dot{q} \end{bmatrix} = \begin{bmatrix} f_{\mathsf{v}}(z, \eta_0, v) \\ 0 \end{bmatrix} & (z, \eta_0, \eta_1, v, q) \in C_{\mathrm{S}} \\ \begin{bmatrix} v^+ \\ q^+ \end{bmatrix} = \begin{bmatrix} \max\{V(z, \eta_0),\ v\} \\ 1 - q \end{bmatrix} & (z, \eta_0, \eta_1, v, q) \in D_{\mathrm{S}} \end{cases}$$

where

$$f_{\mathsf{v}}(z, \eta_0, v) = -\gamma \tanh(v) \sigma_0(z, \eta_0) - \mu\big(v - V_{\mathrm{P}}(z, \eta_0)\big).$$

Theorem: Global Asymptotic Stability

Suppose that

- $\mathcal{A}_{\mathrm{P}}$ is compact,
- Regularity conditions hold (outer semicontinuity, local boundedness of functions, closed sets)
- V_{P} is a Lyapunov function for $\mathcal{H}_{\mathrm{P}} \times \mathcal{H}_{\mathrm{K}0}$ with strict decrease during flows.
- The state of $\mathcal{H}_{\mathrm{K}_1}$ is constrained to compact set

Then, the set $\mathcal{A}_{\mathrm{P}} \times \{0\}$ is (uniformly) asymptotically stable for (z, η_0, z).

Proof Sketch

We introduce a new Lyapunov function

$$V(x) := \max\{V_{\mathrm{P}}(z, \eta_0), v\},$$

then show that it satisfies our relaxed Lyapunov conditions.

Let

$$\sigma_{\mathrm{LSC}}(x) := \begin{cases} \sigma_0(z, \eta_0) & \text{if } V_{\mathrm{P}}(z, \eta_0) > v \\ -f_{\mathsf{v}}(z, \eta_0, v) & \text{if } V_{\mathrm{P}}(z, \eta_0) \leq v \end{cases}$$

We show

- σ_{LSC} is LSC and positive definite with respect to $\mathcal{A}$ on C
- $\dot{V}(x) \leq -\sigma_{\mathrm{LSC}}(x)$.

Proof Sketch

At jumps

$$v^{+} = \max\{V_{\mathrm{P}}(z, \eta_0), v\}.$$

Thus,

$$V(x^{+}) = \max\{V_{\mathrm{P}}(z, \eta_0), v^{+}\} \leq V(x),$$

so $V(x)$ does not increase at jumps.

We apply a hybrid version of our relaxed Lyapunov theorem to get existence of solutions.

Proof Sketch

We prove existence of solutions with unbounded domains:

1. At each point in $C \setminus D$, solutions can flow because
$$F(x) \cap T_C(x) \neq \varnothing.$$
2. The state cannot jump out of $C \cup D$ because
$$G(D) \subset C \cup D.$$

To show that $t \to \infty$ in the domain of each solution:

- For each subsystem, there is a minimum time between sequential jumps, so number of jumps is bounded in finite time.

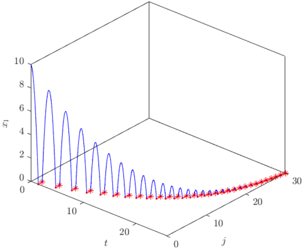
Outline

Software Tools for Hybrid Systems

SHARC

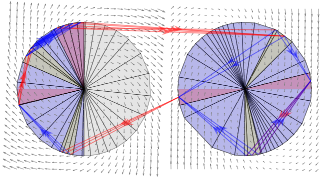
Simulate cyber-physical systems with accurate computational delays.

Hybrid Equations Toolbox



Simulation and plotting of composite hybrid systems

Conical Transition Graphs



Algorithmically check asymptotic stability in hybrid systems.

Outline

Acknowledgements

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- Michael Sommermann.

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- Daniele Venturi
- Alessandro Pinto

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Publications

Wintz and Ricardo G Sanfelice. "Relaxed Lyapunov Conditions". American Control Conference (ACC). 2025.

Wintz, Yasin Sonmez, et al. SHARC: Simulator for Hardware Architecture and Real-time Control". ACM Conference on Hybrid Systems: Computation and Control (HSCC). 2025.

Wintz and Ricardo G. Sanfelice. "Conical Transition Graphs for Analysis of Asymptotic Stability in Hybrid Dynamical Systems". IFAC Conference on Analysis and Design of Hybrid Systems (ADHS). 2024.

Wintz and Ricardo G. Sanfelice. "Forward Invariance-Based Hybrid Control Using Uncertified Controllers". IEEE Conference on Decision and Control (CDC). 2023.

Wintz, Ricardo G. Sanfelice, and João P. Hespanha, "Global Asymptotic Stability of Nonlinear Systems while Exploiting Properties of Uncertified Feedback Controllers via Opportunistic Switching", American Control Conference (ACC). 2022.

Wintz "Optimal Control of a Noncircular Wheel". UC Santa Cruz, MS Thesis. 2020.

Under Review

Wintz and Ricardo G. Sanfelice. “Conical Transition Graphs for Analysis of Asymptotic Stability in Hybrid Dynamical Systems”.

In Preparation

Wintz and Ricardo G. Sanfelice. “Exploiting Uncertified Controllers via Uniting Feedback”.

Wintz and Ricardo G. Sanfelice. “Relaxed Lyapunov Conditions for Non-smooth Dynamical Systems”

Questions?

Outline

SHARC: Simulator for Hardware Architecture and Real-time Control

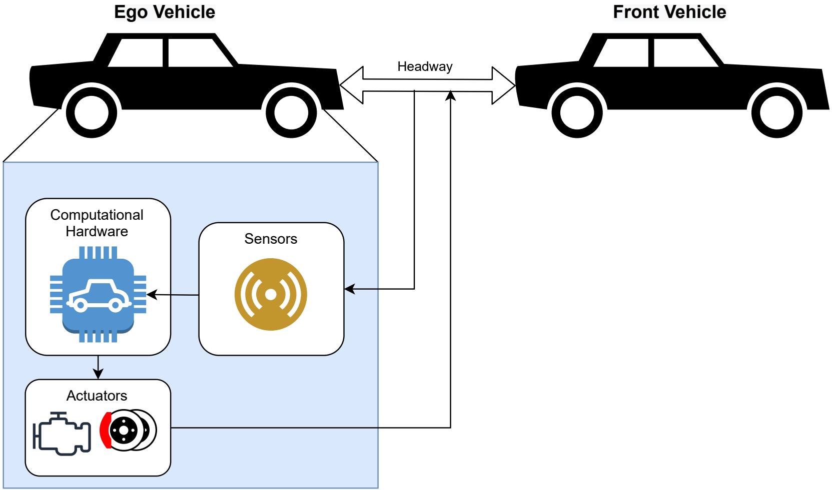
Conical Transition Graph

Technical Definitions

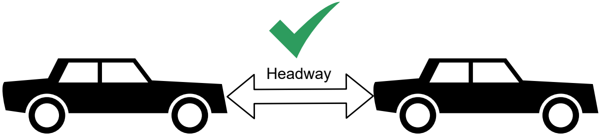
Additional Results for Uniting Feedback

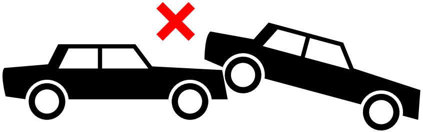
Further Results for Relaxed Lyapunov Conditions

Motivating Example: Adaptive Cruise Control (ACC)



Motivating Example: Adaptive Cruise Control (ACC)





If no computational delays:

⟹ Guaranteed minimum headway

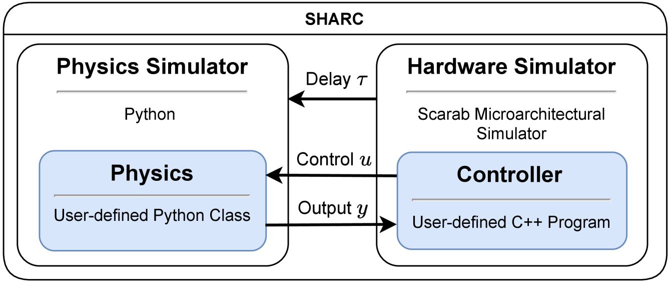
If computational delays:

⟹ ???

Computational delays depend on

- Control Algorithm, implementation, and parameters
- Computational hardware
- Current state and measurements
- Recent computations

Sharc: Simulator for Hardware Architecture and Real-time Control



SHARC

Physics Simulator

Python

Physics

User-defined Python Class

Delay τ

Control u

Output y

Hardware Simulator

Scarab Microarchitectural Simulator

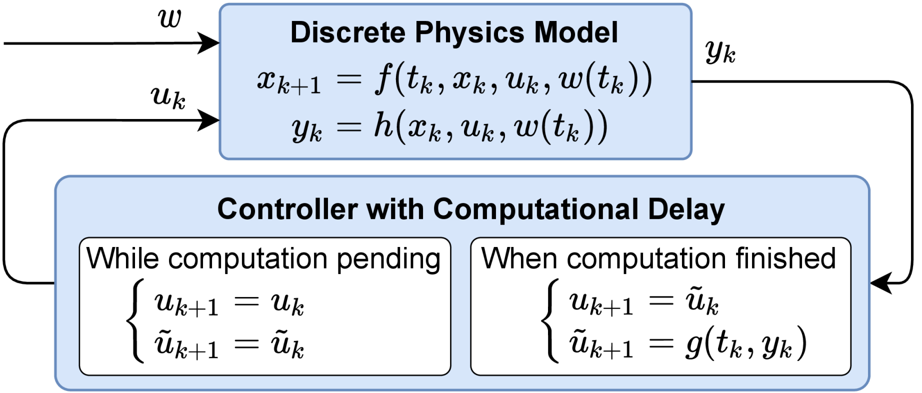
Controller

User-defined C++ Program

Features

- Uses same executable as would be deployed.
- Parallelized to shorten run times.
- Dockerized and easy configuration via JSON files.

Mathematical Model of Delayed Computations



w

u_k

y_k

Discrete Physics Model

$$x_{k+1} = f(t_k, x_k, u_k, w(t_k))$$
$$y_k = h(x_k, u_k, w(t_k))$$

Controller with Computational Delay

While computation pending

$$\begin{cases} u_{k+1} = u_k \\ \tilde{u}_{k+1} = \tilde{u}_k \end{cases}$$

When computation finished

$$\begin{cases} u_{k+1} = \tilde{u}_k \\ \tilde{u}_{k+1} = g(t_k, y_k) \end{cases}$$

Controller Execution Simulation

To estimate controller run time, we use the Scarab Microarchitectural Simulator.

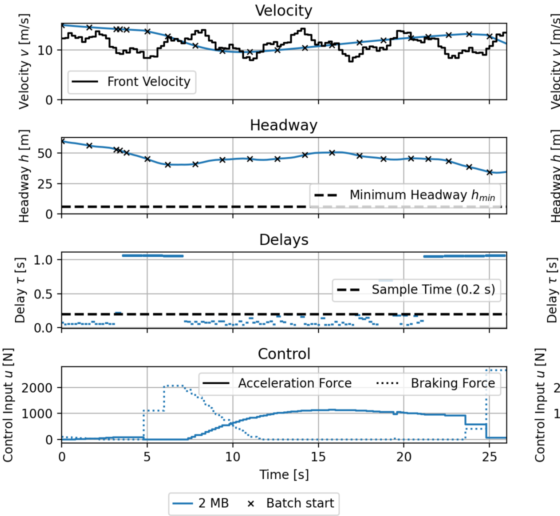
- Low level simulation of controller binary on CPU
- Simulates caching, branch prediction, pipelining, etc.
- Customizable processor parameters
    - Cache size
    - Clock speed
    - Architecture
- Provides detailed statistics.

ACC Example: Instruction Cache Size Comparison

Problem 1 (Linear MPC)

$$\begin{aligned} \text{minimize} \quad & |\textsf{velocity error}|^2 \\ & + |\textsf{control effort}|^2 \\ \text{subject to} \quad & \\ & \textsf{Linear System Dynamics} \\ & \textsf{Linear Safety Constraints} \end{aligned}$$

$\Longrightarrow$ Performance degrades if instruction cache is only 1 KB.



Outline

SHARC: Simulator for Hardware Architecture and Real-time Control

Conical Transition Graph
    Methods for Constructing Conical Transition Graphs

Technical Definitions

Additional Results for Uniting Feedback

Further Results for Relaxed Lyapunov Conditions

Introduction — Problem Setting

Goal: Develop graph-based analysis of asymptotic stability for *conical hybrid systems*

- Allows determining local asymptotic stability of non-conical hybrid systems by using their *conical approximations*.

Introduction — Graph-based Analysis for Hybrid Systems

Previous work using discrete graphs to analyze hybrid systems:

- Asymptotic stability for
  - switched discrete-time linear systems (Philippe et al., 2016)
  - switched discrete-time nonlinear systems (Kundu and Chatterjee, 2016)
  - switched continuous-time linear systems (Langerak and Polderman, 2005)
- Infinite-horizon reachability for linear hybrid automata (Bogomolov et al., 2017).

The present work is (to the best of our knowledge) the first graph-theoretic approach to analyze *asymptotic stability* in *non-switched* hybrid systems.

Introduction — Hybrid Dynamical Systems Framework

$$\mathcal{H} : \begin{cases} \dot{x} &= f(x) \quad x \in C \\ x^+ &= g(x) \quad x \in D \end{cases}$$

- flow set $C \subset \mathbb{R}^n$
- flow map $f : C \to \mathbb{R}^n$

- jump set $D \subset \mathbb{R}^n$
- jump map $g : D \to \mathbb{R}^n$

The continuous-time component of $\mathcal{H} = (C, f, D, g)$ is written

$$(C, f).$$

Introduction — Conical Approximations

Definition 1

Given a hybrid system $\mathcal{H} = (C, f, D, g)$ and a point $x_* \in \overline{C} \cup \overline{D}$ such that $g(x_*) = x_*$, the *conical approximation* of $\mathcal{H}$ at x_* is

$$\begin{cases} f(x) := \text{Constant or linear approximation of } f, & C := \text{Tangent cone of } C \text{ at } x_*, \\ g(x) := \text{Linear approximation of } g, & D := \text{Tangent cone of } D \text{ at } x_*, \end{cases}$$

with each approximation centered at x_*. ◇

Theorem 1 (Goebel and Teel, 2010)

Under sufficient regularity assumptions:

If 0_n is pAS for the conical approximation of $\mathcal{H}$ at x_*, then x_* is pAS for $\mathcal{H}$.

Conical Approximations with Constant Flows

Let $\mathcal{H} = (C, f, D, g)$ be a hybrid system with $x_* \in \overline{C} \cap \overline{D}$ such that

- $g(x_*) = x_*$
- g is continuously differentiable at x_*.
- $f(x_*) \neq 0_n$
- f is continuous at x_*.

Then, the conical approximation of $\mathcal{H}$ at x_* is

$$\mathcal{H}\colon \begin{cases} \dot{x} \; = f(x) := f(x_*) \text{ [constant]}, & C := T_C(x_*), \\ x^+ = g(x) := A_{\mathrm{D}} x \text{ [linear]}, & D := T_D(x_*), \end{cases}$$

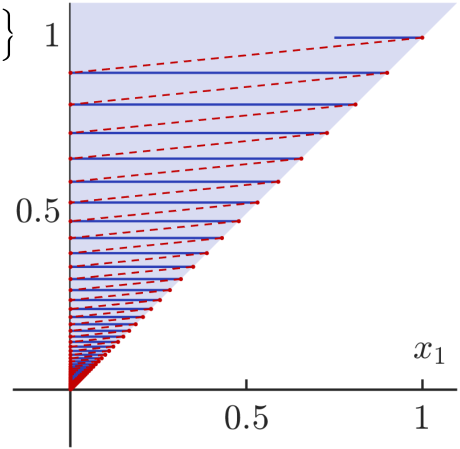
where A_{D} is the Jacobian matrix of g at x_*.

Example 1: Conical Hybrid System

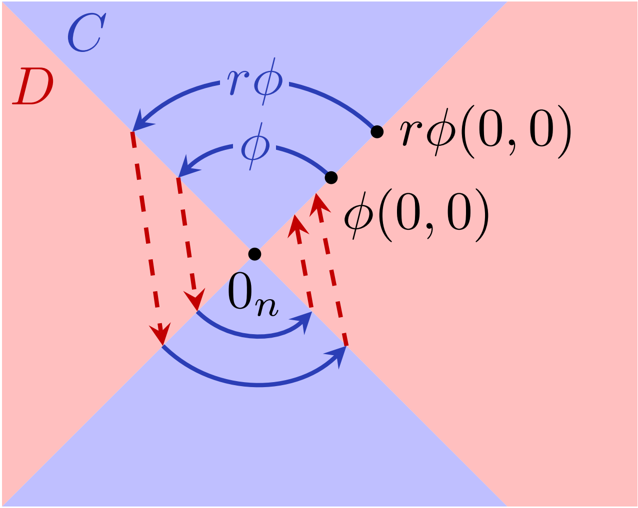
$$\begin{cases} f(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \forall x \in C := \left\{ x \in [0, \infty)^2 \;\middle|\; x_2 \geq x_1 \right\} \\ g(x) := \begin{bmatrix} 0 \\ \gamma x_1 \end{bmatrix} & \forall x \in D := \overline{\mathrm{ray}} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right], \end{cases}$$

with $\gamma > 0$.

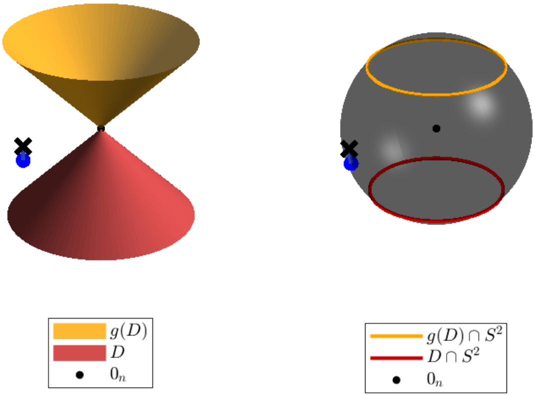
- How to prove the origin asymptotically stable (without a Lyapunov function)?



Introduction — Radial Homogeneity



Introduction — Mapping $\mathbb{R}^n$ to Unit Sphere



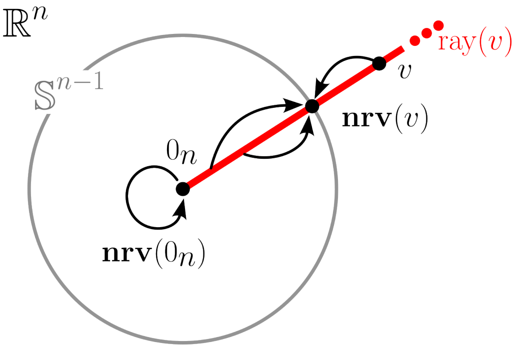
Introduction — Normalized Radial Vectors

The *normalized radial vector* function

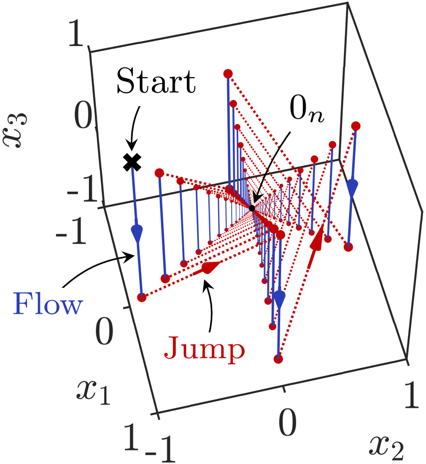
$$\mathrm{nrv} : \mathbb{R}^n \to \mathbb{S}_0^{n-1} := \mathbb{S}^{n-1} \cup \{0_n\}$$

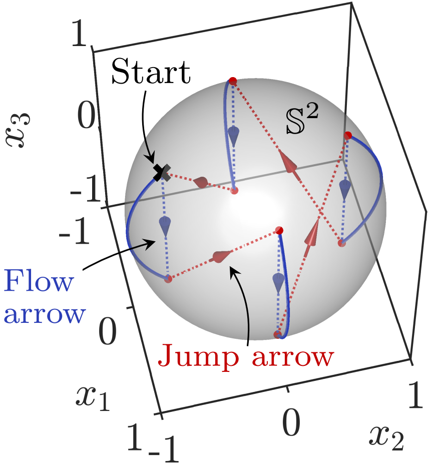
is defined for each $v \in \mathbb{R}^n$ as

$$\mathrm{nrv}(v) := \begin{cases} 0_n & \text{if } v = 0_n \\ \dfrac{v}{|v|} & \text{if } v \neq 0_n. \end{cases}$$



Introduction — Conical Transition Graph (Sketch)





Introduction — Directed Graphs with Weights

A *directed graph* consists of a set of vertices

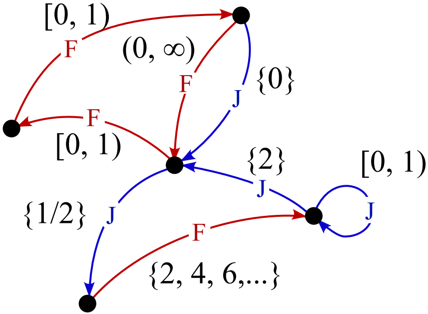
$$\mathcal{V} := \{v_1, v_2, v_3, v_4, v_5\}$$

connected by arrows:

$$\mathcal{A} := \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \text{etc.}\}.$$

Each arrow connects two vertices, e.g.,

$$\mathfrak{a}_1 := v_1 \xrightarrow{\text{F}} v_2, \;\; \mathfrak{a}_2 = v_3 \xrightarrow{\text{J}} v_1, \;\; \text{etc.}$$



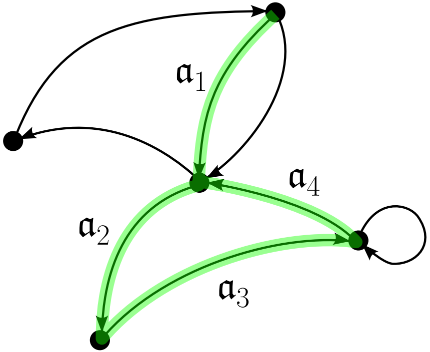
Each arrow is assigned a weight, defined by a *weight function* $\mathcal{W} : \mathcal{V} \rightrightarrows \mathbb{R}$. We write the weight of $\mathfrak{a}$ as $\mathcal{W}(\mathfrak{a}) \subset \mathbb{R}$.

Introduction — Walks Through Graphs

A *walk* w through a graph $\mathcal{G}$ is a sequence of arrows in $\mathcal{A}$:

$$w = (\mathfrak{a}_0,\ \mathfrak{a}_1,\ \ldots,\ \mathfrak{a}_{N-1}) = v_0 \to v_1 \to v_2 \to \cdots \to v_N,$$

such that $\mathfrak{a}_i = v_i \to v_{i+1}$ for each $i = 0, 1, \ldots, N-1$.



Introduction — Walk Weights

The *set-valued weight* of a finite-length walk $w = (\mathfrak{a}_0, \mathfrak{a}_2, \dots, \mathfrak{a}_{N-1})$ is defined as the Minkowski product of the weights

$$\mathcal{W}(w) := \left\{ r_0 \cdot r_1 \cdots r_{N-1} \;\middle|\; r_0 \in \mathcal{W}(\mathfrak{a}_0),\ r_1 \in \mathcal{W}(\mathfrak{a}_1),\ \dots,\ r_{N-1} \in \mathcal{W}(\mathfrak{a}_{N-1}) \right\}$$

For an infinite-length walk $w := (\mathfrak{a}_0, \mathfrak{a}_1, \dots)$, we have that $\mathcal{W}(w) = \{0\}$ if and only if

$$\lim_{m \to \infty} \prod_{k=0}^{m} r_k = 0$$

for every sequence $\{r_k\}_{k=0}^{\infty}$ with $r_k \in \mathcal{W}(\mathfrak{a}_k)$ for all $k \in \mathbb{N}$.

Definition: Conical Transition Graph (CTG)

The *conical transition graph* (CTG) of a conical hybrid system $\mathcal{H} = (C, f, D, g)$ is a weighted, directed graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W}).$$

The set of vertices is defined as

$$\mathcal{V} := (D \cup g(D)) \cap \mathbb{S}_0^{n-1}$$

For each $v^{\ominus} \in \mathcal{V} \cap D$, there is a *jump arrow* from $v^{\ominus}$ to

$$v^{\oplus} = \mathrm{nrv}(g(v^{\ominus}))$$

Definition: Conical Transition Graph (CTG) — Flow Arrows

For each

$$v^{(0)} \in \mathcal{V} \cap g(D) \quad \text{and} \quad v^{(\mathsf{f})} \in \mathcal{V} \cap D,$$

there is a *flow arrow* from $v^{(0)}$ to $v^{(\mathsf{f})}$ if for some $T > 0$, there exists a function

$$[0, T] \ni t \mapsto \xi(t)$$

that satisfies

$$\begin{aligned} &\xi(0) = v^{(0)} \\ &\dot{\xi}(t) = f(\xi(t)) \quad \forall t \in (0, T) \\ &\xi(t) \in C \qquad\quad\;\; \forall t \in (0, T) \\ &\operatorname{nrv}(\xi(T)) = v^{(\mathsf{f})}. \end{aligned} \qquad \text{(Flow Arrow ODE)}$$

Definition: Conical Transition Graph (CTG) — Weights

The weight of each jump arrow $\mathfrak{a}^{\mathrm{J}} = v^{\ominus} \xrightarrow{\mathrm{J}} v^{\oplus}$ is

$$\mathcal{W}(\mathfrak{a}^{\mathrm{J}}) := \big\{ |g(v^{\ominus})| \big\}.$$

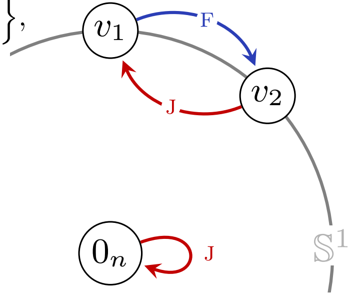
The weight of each flow arrow $\mathfrak{a}^{\mathrm{F}} = v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ is

$$\mathcal{W}(\mathfrak{a}^{\mathrm{F}}) := \big\{ |\xi(T)| \;\big|\; \xi \text{ satisfies the Flow Arrow ODE for some } T > 0 \big\}.$$

Example 1 (Continued): Construction of CTG

$$\mathcal{H}: \begin{cases} f(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \forall x \in C := \left\{ x \in [0, \infty)^2 \;\middle|\; x_2 \geq x_1 \right\}, \\ g(x) := \begin{bmatrix} 0 \\ \gamma x_1 \end{bmatrix} & \forall x \in D := \overline{\mathrm{ray}} \left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \right], \end{cases}$$

with $\gamma > 0$.



Vertices

$$\mathcal{V} = \{0_n, v_1, v_2\}$$

Arrows

$$\mathcal{A} = \{\underbrace{0_n \xrightarrow{\mathrm{J}} 0_n,\ v_2 \xrightarrow{\mathrm{J}} v_1,}_{\text{Jump arrows}}\ \underbrace{v_1 \xrightarrow{\mathrm{F}} v_2}_{\text{Flow arrow}}\}.$$

The weights of the arrows are:

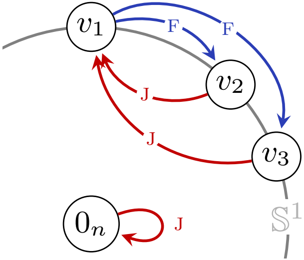
$$\mathcal{W}(0_n \xrightarrow{\mathrm{J}} 0_n) = \{0\}$$
$$\mathcal{W}(v_2 \xrightarrow{\mathrm{J}} v_1) = \{\gamma/\sqrt{2}\}$$
$$\mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_2) = \{\sqrt{2}\}.$$

Example 2

Using f and g from Example 1, consider the flow and jump sets:

$$C' := \left\{ x \in [0,\infty)^2 \;\middle|\; 2x_2 \geq x_1 \right\}$$

$$D' := \overline{\mathrm{ray}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right] \cup \overline{\mathrm{ray}}\left[\begin{smallmatrix}2\\1\end{smallmatrix}\right].$$



The weights of the arrows are:

$$\mathcal{W}(0_n \xrightarrow{\mathrm{J}} 0_n) = \{0\}$$

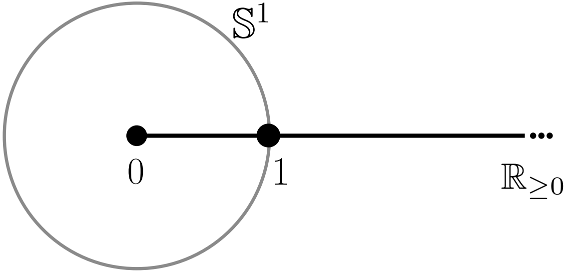
$$\mathcal{W}(v_2 \xrightarrow{\mathrm{J}} v_1) = \{\gamma/\sqrt{2}\} \qquad \mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_2) = \{\sqrt{2}\}$$

$$\mathcal{W}(v_3 \xrightarrow{\mathrm{J}} v_1) = \{2\gamma/\sqrt{5}\} \qquad \mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_3) = \{\sqrt{5}\}.$$

Example 3 — Non-singleton Set-valued Weights

Conical hybrid system:

$$\mathcal{H}:\begin{cases}\dot{x} &= f(x) := -1, \quad C := [0,\infty),\\ x^+ &= g(x) := x/2, \quad D := [0,\infty).\end{cases}$$

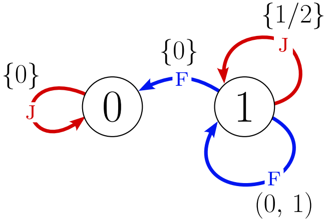


The origin is asymptotically stable for $\mathcal{H}$.

Vertices

$$\mathcal{V} = \{0,\ 1\}$$

Arrows



$$\mathcal{A} = \left\{0 \xrightarrow{\mathrm{J}} 0,\ 1 \xrightarrow{\mathrm{J}} 1,\ 1 \xrightarrow{\mathrm{F}} 0,\ 1 \xrightarrow{\mathrm{F}} 1\right\}.$$

Main Result

Theorem 2

Let $\mathcal{H} = (C, f, D, g)$ be a conical hybrid system with conical transition graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. Suppose the following:

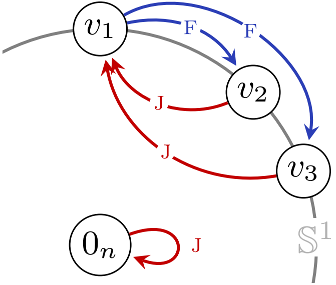
1. *The origin is pre-asymptotically stable for (C, f).*
2. *There exists $M > 0$ such that every walk w through $\mathcal{G}$ satisfies $\sup \mathcal{W}(w) \leq M$.*
3. *Every well-formed infinite-length walk w through $\mathcal{G}$ satisfies $\mathcal{W}(w) = \{0\}$.*

Then, the origin of $\mathcal{H}$ is pAS.

When $\mathcal{V}$ is finite:

- Condition 2 is implied by 3 if each individual arrow weight is bounded.
- Condition 3 is satisfied if and only if $\sup \mathcal{W}(w) < 1$ for every well-formed elementary cycle w in $\mathcal{G}$.

Example 2 — Continued



Arrow Weights:

$$\mathcal{W}(0_n \xrightarrow{\text{J}} 0_n) = \{0\}$$

$$\mathcal{W}(v_2 \xrightarrow{\text{J}} v_1) = \{\gamma/\sqrt{2}\} \qquad \mathcal{W}(v_1 \xrightarrow{\text{F}} v_2) = \{\sqrt{2}\}$$

$$\mathcal{W}(v_3 \xrightarrow{\text{J}} v_1) = \{2\gamma/\sqrt{5}\} \quad \mathcal{W}(v_1 \xrightarrow{\text{F}} v_3) = \{\sqrt{5}\}.$$

Weights of Elementary Cycles:

$$\mathcal{W}(0_n \xrightarrow{\text{J}} 0_n) = \{0\}$$

$$\mathcal{W}(v_1 \xrightarrow{\text{F}} v_2 \xrightarrow{\text{J}} v_1) = \mathcal{W}(v_2 \xrightarrow{\text{J}} v_1 \xrightarrow{\text{F}} v_2) = \{\gamma\}$$

$$\mathcal{W}(v_1 \xrightarrow{\text{F}} v_3 \xrightarrow{\text{J}} v_1) = \mathcal{W}(v_3 \xrightarrow{\text{J}} v_1 \xrightarrow{\text{F}} v_3) = \{2\gamma\}$$

$\Longrightarrow$ For all $0 < \gamma < 1/2$, the origin is pAS.

Future Work

Directions for future work:

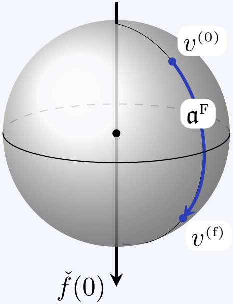
1. Expand the scope of systems for which our approach is tractable.
   - Construct CTG's for conical hybrid systems with linear flows.
   - Handle CTG's that have a large or infinite number of vertices.
2. Extend the CTG results to more general hybrid systems:
   - Hybrid systems with switching between logical modes.
   - Hybrid systems with set-valued flow and jump maps.

Proposition 1 (Flow Arrows — Constant Flows)

Let $\mathcal{H}$ be a conical system with constant flows and let $\mathcal{G}$ be the CTG of $\mathcal{H}$.

Then, $v^{(0)} \xrightarrow{\mathrm{F}} v^{(f)}$ is a flow arrow in $\mathcal{G}$ from $v^{(0)} \in \mathcal{V} \cap g(D)$ to $v^{(f)} \in \mathcal{V} \cap D$ if and only if

- $\operatorname{nrv}(v_{\perp}^{(0)}) = \operatorname{nrv}(v_{\perp}^{(f)})$,
- $v^{(f)} \neq 0_n$ or $v^{(0)} \neq 0_n$,
- $\langle v^{(f)} - v^{(0)},\, f(0_n)\rangle \geq 0$,
- $v_{\perp}^{(0)} \neq 0_n \implies \langle v^{(f)} - v^{(0)},\, f(0_n)\rangle > 0$,
- $\theta v^{(0)} + (1-\theta) v^{(f)} \in C \quad \forall \theta \in [0,1]$.



A formula for the weight of $\mathfrak{a}^{\mathrm{F}} := v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ is given in the paper.

Computational Considerations for Finite Conical Transition Graphs

A walk through a graph is called an *elementary cycle* if it starts and ends at the same vertex and does not visit any other vertex more than once.

We can enumerate over all of the elementary cycles using Johnson's enumeration algorithm (Johnson, 1975). The worst-case time complexity of Johnson's algorithm is

$$O\big((\text{no. of vertices} + \text{no. of edges})(\text{no. of elementary cycles} + 1)\big).$$

Outline

SHARC: Simulator for Hardware Architecture and Real-time Control

Conical Transition Graph

Technical Definitions
- Non-smooth Analysis
- Set-valued Lie derivative

Additional Results for Uniting Feedback

Further Results for Relaxed Lyapunov Conditions

Definition of UGAS

Definition 2

A nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *uniformly globally stable* if there exists a continuous, strictly increasing function α such that every solution x to $\mathcal{H}$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \alpha(|x(0,0)|_{\mathcal{A}})$ for each $(t,j) \in \operatorname{dom} x$; and
- *uniformly globally attractive* for $\mathcal{H}$ if every maximal solution is complete and for all $\varepsilon > 0$ and $r > 0$, there exists $T > 0$ such that every solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq r$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$ such that $t + j \geq T$.
- If $\mathcal{A}$ is both uniformly globally stable and uniformly globally attractive for $\mathcal{H}$, then it is said to be *uniformly globally asymptotically stable* (UGAS) for $\mathcal{H}$.

◇

Because κ_0 is Lyapunov-certified there exists a Lyapunov function

$$V : \mathbb{R}^{n_{\mathrm{P}}} \to [0, \infty)$$

that guarantees $\mathcal{A}$ is UGAS for

$$\dot{z} = f_{\mathrm{P}}(z, \kappa_0(z)).$$

Namely, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ such that

$$\begin{aligned} \alpha_1(|z|_{\mathcal{A}}) \leq V(z) &\leq \alpha_2(|z|_{\mathcal{A}}) && \forall z \in \mathbb{R}^n, \\ \dot{V}_0(z) &\leq -\rho(|z|_{\mathcal{A}}) && \forall z \in \mathbb{R}^n. \end{aligned}$$

Hybrid Systems

We consider hybrid systems modeled as

$$\mathcal{H} : \begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D \end{cases}$$

with

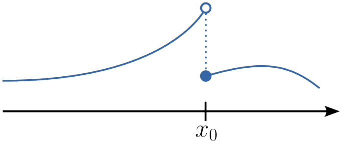
- flow set $C \subset \mathbb{R}^n$
- flow map $f : C \to \mathbb{R}^n$

- jump set $D \subset \mathbb{R}^n$
- jump map $g : D \to \mathbb{R}^n$

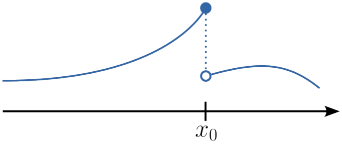
Definition (Lower Semicontinuous)

A function f is *lower semicontinuous* (LSC) at x_0 if

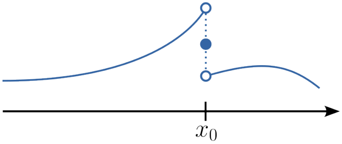
$$f(x_0) \leq \liminf_{x \to x_0} f(x).$$



Lower Semicontinuous (LSC).



Upper Semicontinuous (USC).



Not LSC, not USC.

Contingent Cone

Let $S \subset \mathbb{R}^n$ be nonempty and let $x \in \overline{S}$. The contingent cone [8] to S at x is denoted $T_S(x)$, and is given by The *contingent cone* $T_S(x)$ is the set of all vectors $v \in \mathbb{R}^n$ such that there exist a sequence of positive real numbers $h_i \to 0^+$ and a sequence of vectors $v_i \to v$ such that

$$x + h_i v_i \in S$$

for all $i \in \mathbb{N}$.

Clark Generalized Gradient and Directional Derivative

For a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$, the *Clarke generalized gradient* of V at $x \in \mathbb{R}^n$ is

$$\partial^\circ V(x) := \operatorname{conv}\left\{ \lim_{i\to\infty} \nabla V(x_i) \;\middle|\; \exists (x_i \to x) \text{ s.t. } V \text{is differentiable at each } x_i \right\}. \tag{2}$$

The *Clarke generalized directional derivative* of V at x in the direction $w \in \mathbb{R}^n$ is given by $V^\circ(x, w) = \max_{\zeta \in \partial^\circ V(x)} \langle \zeta,\, w \rangle$. The *Clarke generalized directional derivative* of V at $x \in \mathbb{R}^n$ in the direction $w \in \mathbb{R}^n$ is given by

$$V^\circ(x, w) = \max_{\zeta \in \partial^\circ V(x)} \langle \zeta,\, w \rangle. \tag{3}$$

Lie Derivative

For a differential inclusion $\dot{x} \in F(x)$ with a set-valued map $F : \mathbb{R}^n \rightrightarrows \mathbb{R}^n$ and a Lipschitz continuous function $V : \mathbb{R}^n \to \mathbb{R}$, we define the *set-value Lie derivative* of V along F as

$$\mathcal{L}_F V(x) := \sup\{\langle \zeta,\, f \rangle \mid \zeta \in \underbrace{\partial^\circ V(x)}_{\text{Generalized Gradient}},\ f \in F(x) \cap \underbrace{T_C(x)}_{\text{Tangent Cone}}\}.$$

Furthermore, since $\mathcal{L}_F V(x)$ is a set in $\mathbb{R}$, we can write the least upper bound as $\sup \mathcal{L}_F V(x)$.

Rate of change of V for $\dot{x} \in F(x)$

Consider

$$\dot{x} \in F(x) \quad x \in C \subset \mathbb{R}^n. \tag{$\star$}$$

and a locally Lipschitz function $V : \mathbb{R}^n \to \mathbb{R}$.

For each $x \in C$,

$$\dot{V}(x) := \mathcal{L}_F V(x) = \sup\{\langle \zeta,\, f\rangle \mid \zeta \in \underbrace{\partial^\circ V(x)}_{\text{Generalized Gradient}},\ f \in F(x) \cap \underbrace{T_C(x)}_{\text{Tangent Cone}}\}.$$

For any solution ϕ to $(\star)$ and all $t_1, t_2 \in \operatorname{dom}\phi$,

$$V\big(\phi(t_2)\big) - V\big(\phi(t_1)\big) \leq \int_{t_1}^{t_2} \dot{V}\big(\phi(t)\big)\, dt.$$

Outline

Theorem (Forward Invariance Without Chattering)

Suppose that

- B *is a* $\mathcal{C}^1$ *barrier function of* K *for* $\dot{z} = f_0(z)$.
- f_0 *and* f_1 *are globally Lipschitz continuous with Lipschitz constants* L_0 *and* L_1.
- δ_0, δ_1, θ_0, *and* θ_1 *are continuous and satisfy the threshold function inequalities.*
- *There exists* $\tau > 0$ *such that for all* $z^0 \in \mathcal{Z}_{0\mapsto 1}$ *and* $z^1 \in \mathcal{Z}_{1\mapsto 0}$,

$$|z^0 - z^1| \geq \tau \max\{|f_0(z^0)| \exp(L_0\tau),\ |f_1(z^1)| \exp(L_1\tau)\}.$$

Then,

- τ *is a lower bound on the time between jumps for all solutions to* $\mathcal{H}$.
- *Every maximal solution* ϕ *to* $\mathcal{H}$ *is complete and* $\sup_t \operatorname{dom} \phi = \infty$.

Proof Sketch of Section 1

- Forward pre-invariance is proven using the same barrier function as in Theorem 1.
- Solutions to $\dot{z} = f_0(z)$ and $\dot{z} = f_1(z)$ cannot escape to infinity in finite time because f_0 and f_1 are globally Lipschitz.
- Let $z^0 \in \mathcal{Z}_{0\mapsto 1}$ and $z^1 \in \mathcal{Z}_{1\mapsto 0}$. To prove τ is a lower bound on the time between jumps, we show
  - The (unique) solution to $\dot{z} = f_0(z)$ starting at z^0 satisfies

$$|z^0 - z^1| \geq \tau|f_0(z^0)|\exp(L_0\tau) \geq |z(t) - z^0| \quad \forall t \in [0, \tau].$$

Thus, in time τ, a solution to $\mathcal{H}$ cannot move from $(z^0, 1) \in g(D_{0\mapsto 1})$ to $(z^1, 1) \in D_{1\mapsto 0}$.

Proof Sketch of Section 1

- The (unique) solution to $\dot{z} = f_1(z)$ starting at z^1 satisfies

  $$|z^0 - z^1| \geq \tau| f_1(z^1)| \exp(L_1 \tau) \geq |z(t) - z^1| \quad \forall t \in [0, \tau].$$

  Thus, in time τ, a solution to $\mathcal{H}$ cannot move from $(z^1, 0) \in g(D_{1 \mapsto 0})$ to $(z^0, 0) \in D_{0 \mapsto 1}$.
- The time to move from $g(D)$ to D is at least τ.

Example: Lower Bound on Switching Times

Consider the plant

$$\dot{z} = f_{\mathrm{P}}(z, u) := \begin{bmatrix} z_1 \\ u \end{bmatrix}$$

with $z = (z_1, z_2) \in \mathbb{R}^2$ and $u \in \mathbb{R}$.

Admissible Set: Lower Half Plane of $\mathbb{R}^2$

Controllers: $\kappa_0(z) := -|z_1|$

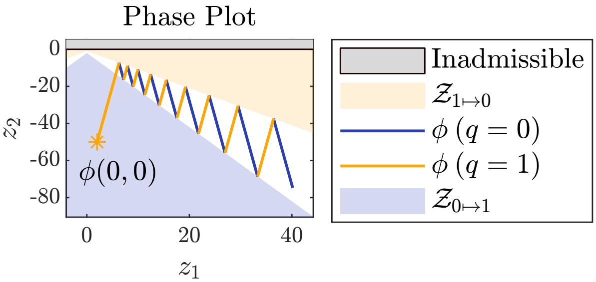
$\kappa_1(z) := +|z_1|$

Barrier Function: $B(z) := z_2$.

Thresholds: $\delta_0(z) := -2 - 2|z_1|$

$\delta_1(z) := -1 - |z_1|$.

Satisfies Theorem 1 $\implies$ K is forward invariant.



Solutions are unbounded $\implies$ Theorem 1 does not guarantee solutions existence for all $t > 0$.

Example: Lower Bound on Switching Times

We find that for $\tau := 0.25$,

$$|z^0 - z^1| \geq \frac{|z_1^0| + 1}{\sqrt{5}} > \tau | f_0 (z^0)| \exp(L_0 \tau),$$
$$|z^0 - z^1| \geq \frac{|z_1^1| + 1}{\sqrt{2}} > \tau | f_1 (z^1)| \exp(L_1 \tau).$$

$$\text{Satisfies Section 1} \implies \begin{cases} \text{The time between jumps is at least } \tau = 0.25. \\ \text{Every maximal solution exists for all } t \geq 0. \end{cases}$$

Clarke Generalized Gradient of $\widetilde{V}(x)$

For the function

$$\widetilde{V}(x) := \max\{V(z), v\},$$

the Clarke generalized gradient at $x = (z, v, q)$ in the direction $w = (w_z, w_v, 0)$ is

$$\widetilde{V}^\circ(x, w) = \begin{cases} \langle \nabla_z V(z),\, w_z \rangle & \text{if } V(z) > v, \\ \max\{\langle \nabla_z V(z),\, w_z \rangle, w_v\} & \text{if } V(z) = v, \\ w_v & \text{if } V(z) < v. \end{cases} \tag{4}$$

Outline

SHARC: Simulator for Hardware Architecture and Real-time Control

Conical Transition Graph

Technical Definitions

Additional Results for Uniting Feedback

Further Results for Relaxed Lyapunov Conditions
- Insertion Theorems
- Simplified Conditions on Hybrid Time Domains

$\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)$.

Given σ_{LSC}.

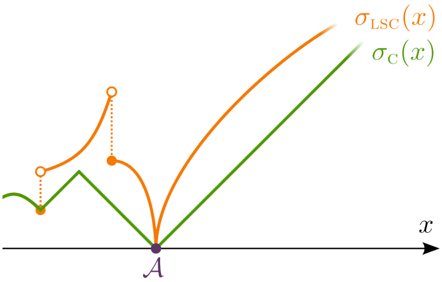
For all $x \in \mathbb{R}^n$, let

$$\sigma_{\mathrm{C}}(x) := \inf_{x' \in \mathbb{R}^n} \big(\sigma_{\mathrm{LSC}}(x') + \ell|x' - x|\big).$$

Then,

- σ_{C} is Lipschitz continuous with Lipschitz constant ℓ
- σ_{C} is positive definite w.r.t. $\mathcal{A}$
- $\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)$ for all $x \in \mathbb{R}^n$.

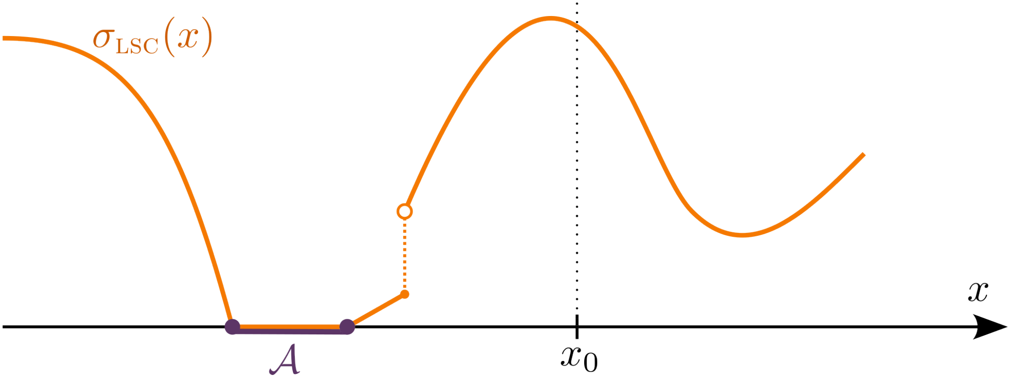
$\sigma_{\mathrm{LSC}}(x) \geq \sigma_{\mathrm{C}}(x)$



Construction of σ_{C}

Given $x_0 \in \operatorname{dom}(\sigma_{\mathrm{LSC}})$,

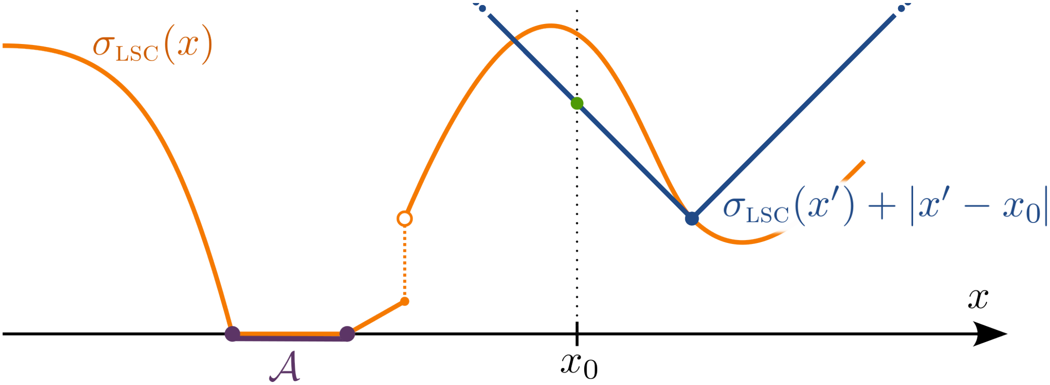
$$\sigma_{\mathrm{C}}(x_0) := \inf_{x'} \big(\sigma_{\mathrm{LSC}}(x') + |x' - x_0|\big).$$



Construction of σ_{C}

Given $x_0 \in \operatorname{dom}(\sigma_{\mathrm{LSC}})$,

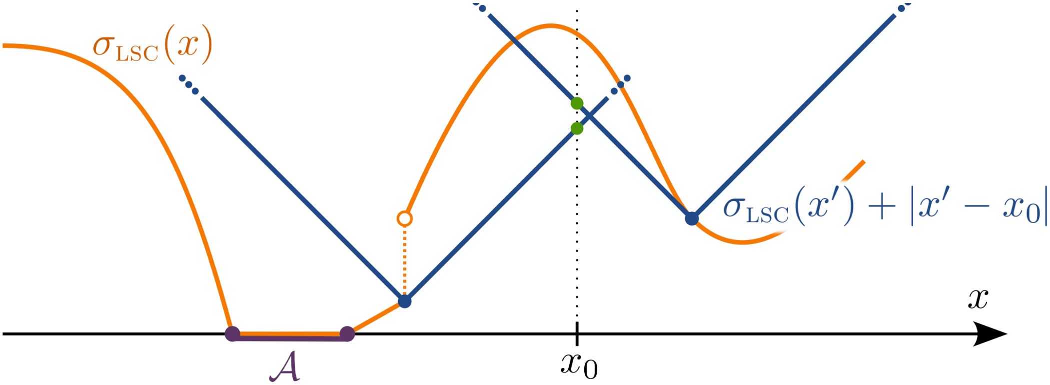
$$\sigma_{\mathrm{C}}(x_0) := \inf_{x'}\left(\sigma_{\mathrm{LSC}}(x') + |x' - x_0|\right).$$



Construction of σ_{C}

Given $x_0 \in \mathrm{dom}(\sigma_{\mathrm{LSC}})$,

$$\sigma_{\mathrm{C}}(x_0) := \inf_{x'}\big(\sigma_{\mathrm{LSC}}(x') + |x' - x_0|\big).$$



Construction of σ_{C}

Given $x_0 \in \operatorname{dom}(\sigma_{\mathrm{LSC}})$,

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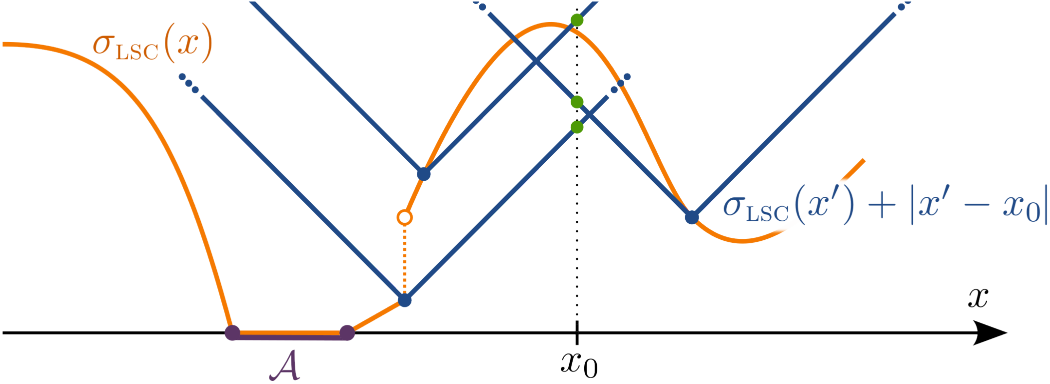

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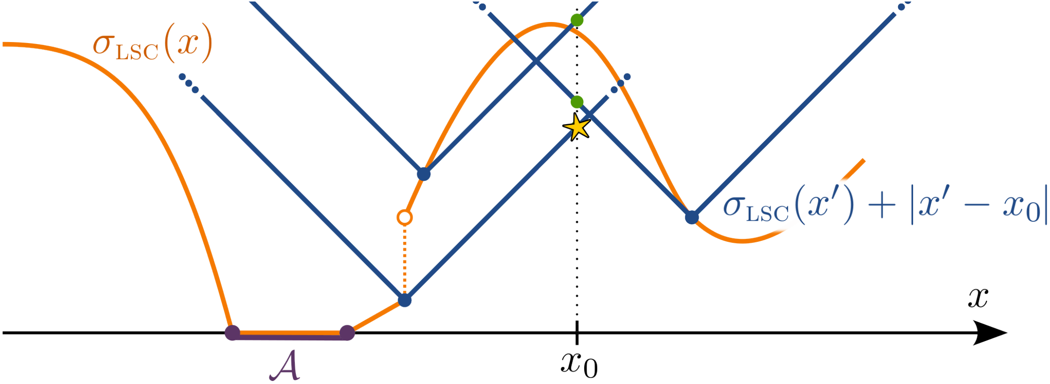

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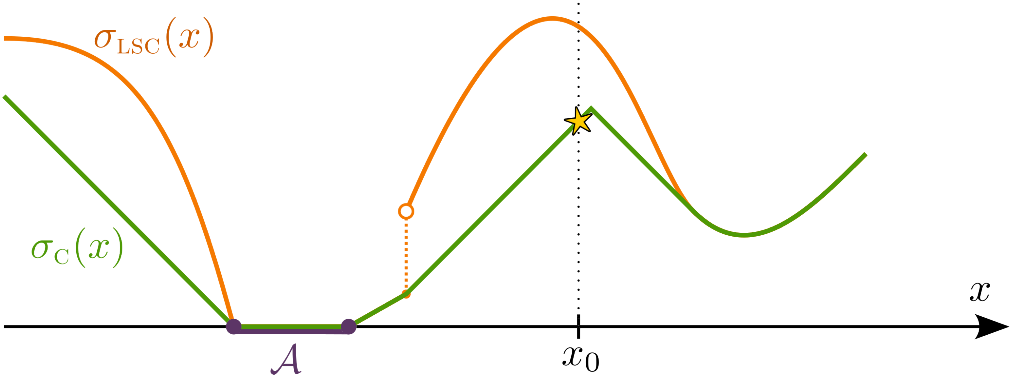
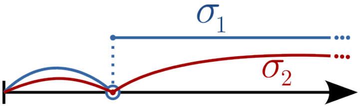


Non-Example

Consider

$$\sigma_1(x) := \begin{cases} x(1-x) & \text{if } x \in [0,1) \\ 1 & \text{if } x \geq 1 \end{cases}$$



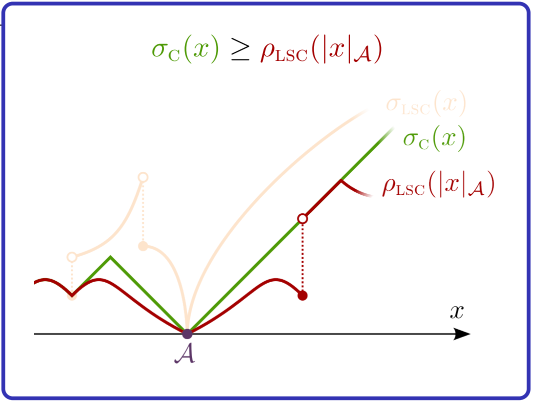
- ✓ Positive definite (w.r.t. 0).
- ✗ Not lower semicontinuous at $x = 1$.

Any continuous function σ_2 between 0 and σ_1 is 0 at $x = 1$ because

$$\liminf_{x \to 1} \sigma_1(x) = 0,$$

- ✓ Continuous
- ✗ Not Positive definite (w.r.t. 0)

$\sigma_{\mathrm{C}}(x) \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}})$.



Given σ_{C}.

Let

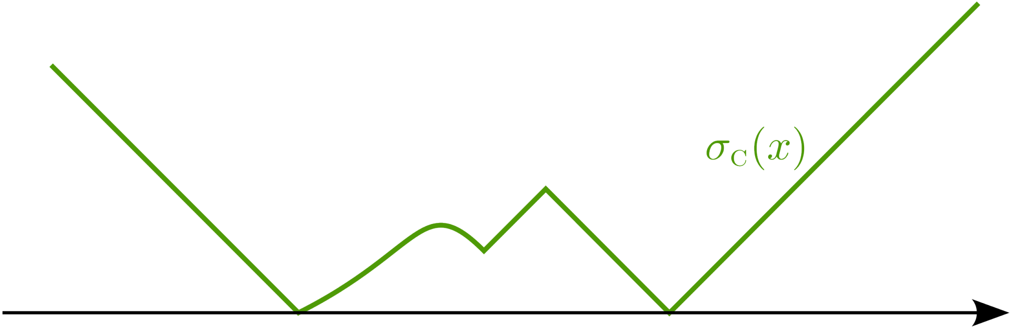
$$\rho_{\mathrm{LSC}}(r) := \inf\{\sigma_{\mathrm{C}}(x) : |x|_{\mathcal{A}} = r\} \ \ \forall r \geq 0.$$

Then,

- ρ_{LSC} is lower semicontinuous
- ρ_{LSC} is positive definite (w.r.t. 0)
- $\sigma_{\mathrm{C}}(x) \geq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.

Construction of σ_{C}

$$\rho_{\mathrm{LSC}}(r) := \inf\{\sigma_{\mathrm{C}}(x) : |x|_{\mathcal{A}} = r\} \quad \forall r \geq 0.$$



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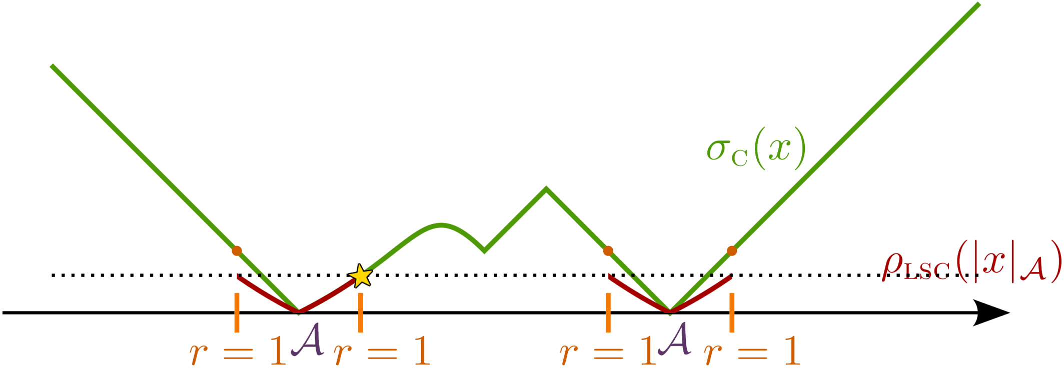

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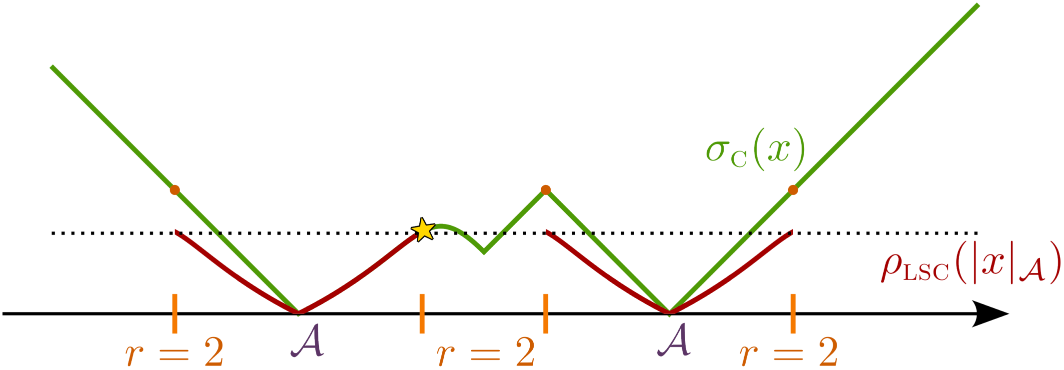
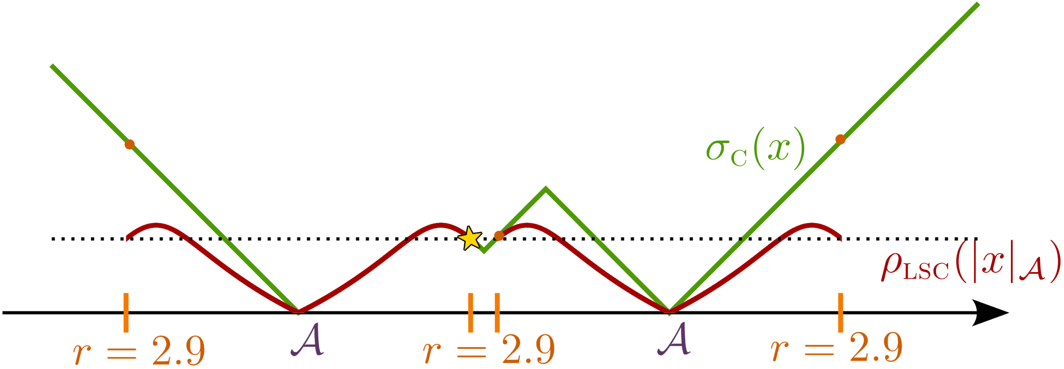


Construction of σ_{C}

$$\rho_{\mathrm{LSC}}(r) := \inf\left\{\sigma_{\mathrm{C}}(x) : |x|_{\mathcal{A}} = r\right\} \quad \forall r \geq 0.$$



Construction of σ_{C}

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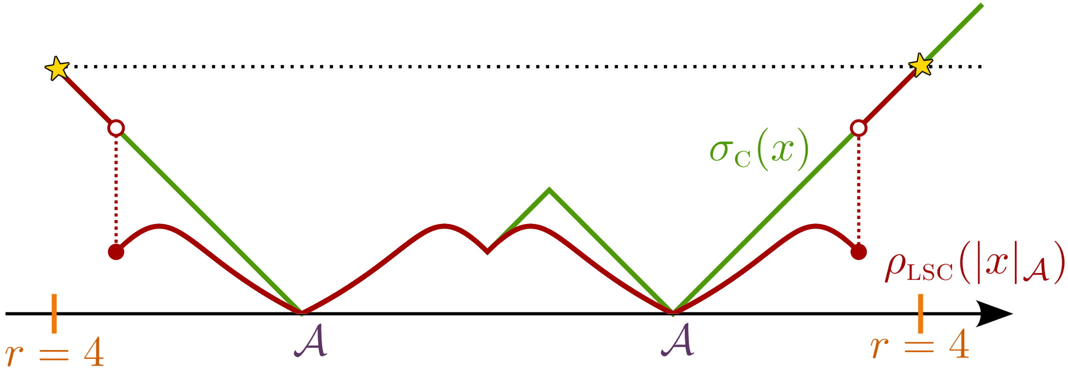


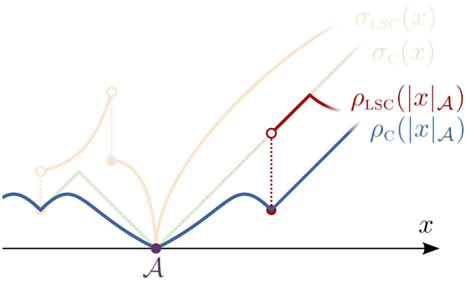
$\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$

Given ρ_{LSC}.

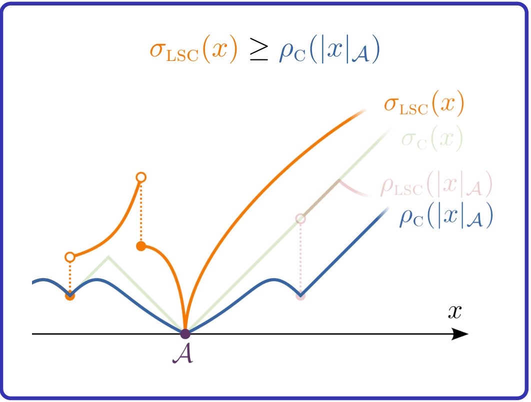
There exists ρ_{C} such that

- ρ_{C} is Lipschitz continuous
- ρ_{C} is positive definite w.r.t. 0
- $\rho_{\mathrm{LSC}}(s) \geq \rho_{\mathrm{C}}(s)$ for all $s \geq 0$.

$\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$



$\sigma_{\mathrm{LSC}}(x) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$



Proposition 2

Suppose $\mathcal{A}$ is compact and $\sigma_{\mathrm{LSC}} : \mathbb{R}^n \to [0, \infty)$ is

- *lower semicontinuous, and*
- *positive definite w.r.t. $\mathcal{A}$.*

Then, there exists ρ_{C} such that

- *ρ_{C} is Lipschitz continuous,*
- *ρ_{C} is positive definite w.r.t. 0, and*
- *$\sigma_{\mathrm{LSC}}(x) \geq \rho_{\mathrm{C}}(|x|_{\mathcal{A}})$ for all $x \in \mathbb{R}^n$.*

Proof Sketch

Suppose $\sigma_{\mathrm{LSC}} \in \mathcal{PD}(\mathcal{A})$ is LSC. By **??**, there exists a continuous function $\sigma_{\mathrm{C}} \in \mathcal{PD}(\mathcal{A})$ such that

$$\sigma_{\mathrm{C}}(x) \leq \sigma_{\mathrm{LSC}}(x) \quad \forall x \in \mathbb{R}^n.$$

By **??**, there exists an LSC and positive definite function $\rho_{\mathrm{LSC}} \in \mathcal{PD}(0)$ such that

$$\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \leq \sigma_{\mathrm{C}}(x) \quad \forall x \in \mathbb{R}^n.$$

Again, by **??**, for any $\ell > 0$ there exists an ℓ-Lipschitz continuous function $\rho_{\mathrm{C}} \in \mathcal{PD}(0)$ such that

$$\rho_{\mathrm{C}}(r) \leq \rho_{\mathrm{LSC}}(r) \quad \forall r \geq 0.$$

Thus, for all $x \in \mathbb{R}^n$,

$$\rho_{\mathrm{C}}(|x|_{\mathcal{A}}) \leq \rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \leq \sigma_{\mathrm{C}}(x) \leq \sigma_{\mathrm{LSC}}(x).$$

Proposition (Simplified Conditions for Persistent Flows).

Consider a hybrid system $\mathcal{H}$ and a nonempty closed set $\mathcal{A}$. Suppose that for each $r \geq 0$, there exist $\Delta_T > 0$ and $\Delta_J > 0$ such that for every solution ϕ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$ and for every $(t_0, j_0), (t_1, j_1) \in \operatorname{dom} \phi$,

$$|t_1 - t_0| \leq \Delta_T \implies |j_1 - j_0| \leq \Delta_J. \tag{5}$$

Then, for each $r \geq 0$, there exist $N_r \geq 0$ and $\gamma_r \in \mathcal{K}_\infty$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$t \geq \gamma_r(t + j) - N_r \quad \forall (t, j) \in \operatorname{dom} \phi. \tag{6}$$

Proposition (Simplified Conditions for Persistent Jumps).

Consider a hybrid system $\mathcal{H}$ and a nonempty closed set $\mathcal{A}$. Suppose that for each $r \geq 0$, there exists $\Delta_T > 0$ and $\Delta_J > 0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$ and for all $(t_0, j_0), (t_1, j_1) \in \operatorname{dom} \phi$,

$$|j_1 - j_0| \leq \Delta_J \implies |t_1 - t_0| \leq \Delta_T. \tag{7}$$

Then, for each $r > 0$, there exist $\gamma_r \in \mathcal{K}_\infty$ and $N_r \geq 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \in (0, r]$,

$$j \geq \gamma_r(t + j) - N_r \quad \forall (t, j) \in \operatorname{dom} \phi. \tag{8}$$

Proof that $\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) \leq \sigma_{\mathrm{C}}(x)$.

For any $x \in \mathbb{R}^n$,

$$\rho_{\mathrm{LSC}}(|x|_{\mathcal{A}}) = \inf\left\{\sigma_{\mathrm{C}}(x') : |x|_{\mathcal{A}} = |x'|_{\mathcal{A}}\right\} \leq \sigma_{\mathrm{C}}(x).$$

$\square$

Proof that ρ_{LSC} is positive definite.

For each $r \geq 0$, σ_{C} attains a minimum on the compact set $\{x : |x|_{\mathcal{A}} = r\}$.

The minimum is positive if and only if $r > 0$ since σ_{C} is positive definite w.r.t. $\mathcal{A}$.

Therefore, ρ_{LSC} is positive definite (w.r.t. 0). ☐

Proof sketch that ρ_{LSC} is lower semicontinuous.

To establish that ρ_{LSC} is LSC, we exploit the fact that $\mathcal{A}$ is compact and σ_{C} is continuous. For each $r \geq 0$, we pick a compact set K_r containing an open neighborhood of $\mathcal{A} + r\mathbb{B}$. Since σ_{C} is continuous, its restriction to the compact set K_r is uniformly continuous. This allows us to do a δ-ϵ proof of lower semicontinuity. ☐