Global Asymptotic Stability of Nonlinear Systems while Exploiting Properties of Uncertified Feedback Controllers via Opportunistic Switching

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Outline

Introduction – Switched Controllers

Sometimes, a single continuous controller cannot satisfy design requirements.

Switching between multiple controllers has been used to...

- Achieve robust global asymptotic stability around obstructions.[1]
- Unite multiple Lyapunov-certified controllers (such as local and global controllers) to achieve global asymptotic stability.[2]
- Provide a backup controller that guarantees constraint safety when the primary controller is not provably safe.[3]

[1] Mayhew, Sanfelice, and Teel (2011) and Sanfelice, Messina, et al. (2006).
[2] Prieur (2001), Teel and Kapoor (1997), and El-Farra, Mhaskar, and Christofides (2005).
[3] Seto et al. (1998).

Why Use an Uncertified Controller?

An uncertified controller may have “better” properties compared to available certified controllers:

- More energy efficient
- Faster convergence
- Requires less computation

Examples:

- Linear quadratic regulator (LQR) for the linearization of a system with an unknown basin of attraction.
- Model predictive control (MPC) that occasionally fails to compute an update.
- Black box controllers (e.g., neural network controllers).

Problem Setting

Consider a continuous-time plant

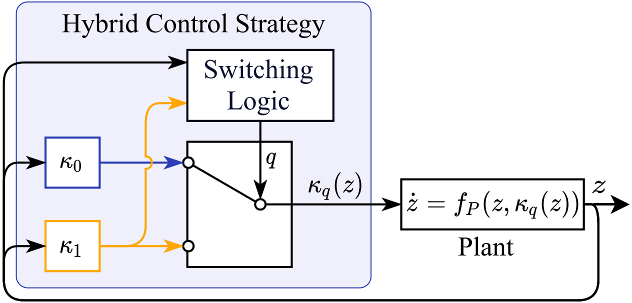
$$\dot{z} = f_P(z, u), \quad z \in \mathbb{R}^n, u \in \mathbb{R}^m.$$

Our goal is to render a compact set $\mathcal{A} \subset \mathbb{R}^n$ globally asymptotically stable (GAS).

Given two continuous controllers:

κ_0: a Lyapunov-certified controller

κ_1: any continuous controller



Our contribution:

Switching logic for $q \in \{0, 1\}$ such that

- $\mathcal{A}$ is GAS
- κ_1 is preferred over κ_0

Problem Setting – Lyapunov-certified Controller κ_0

Because κ_0 is Lyapunov-certified there exists a Lyapunov function

$$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

that guarantees $\mathcal{A}$ is GAS for

$$\dot{z} = f_P(z, \kappa_0(z)).$$

Example: Model Predictive Controller with Slow Computation

Consider a nonlinear plant

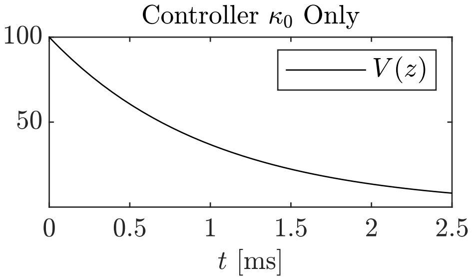
$$\dot{z} = f_P(z, u)$$

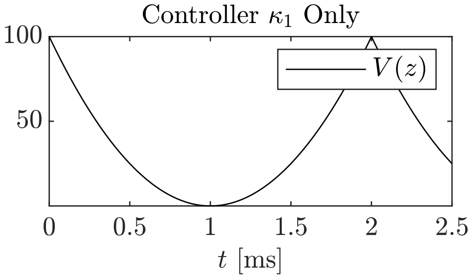
and two controllers:

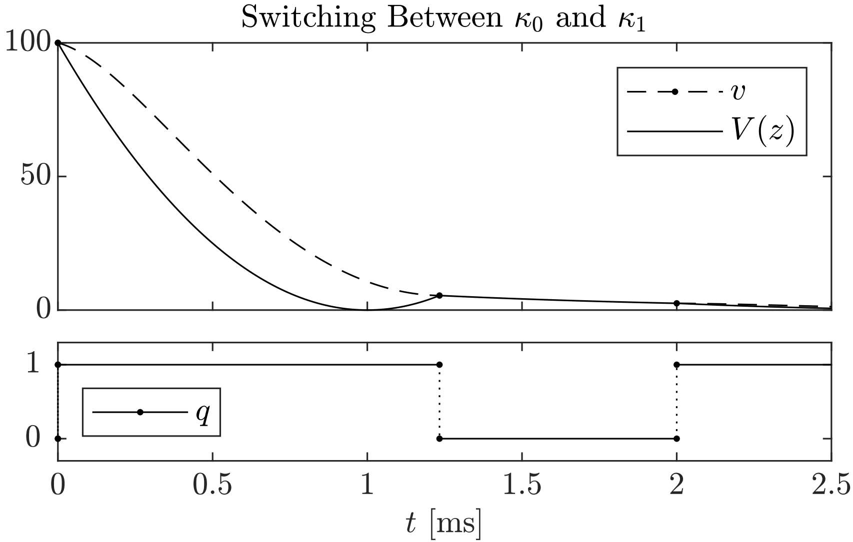
- κ_0: Lyapunov-certified controller
- κ_1: Model predictive controller (MPC) with a sampling period of 1 ms

Suppse A new MPC feedback value is not available at every sample time.

When should we switch?







- The dynamics of v are described later.

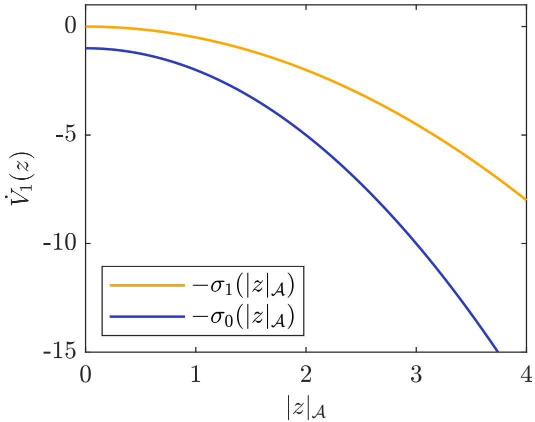
Hybrid Control Strategy – Switching Logic

Let $\dot{V}_1(z) := \langle \nabla V(z), f_P(z, \kappa_1(z)) \rangle$.

Threshold functions:

Let $\sigma_0, \sigma_1 : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be continuous functions such that

- σ_1 is positive definite
- $\sigma_0(s) > \sigma_1(s)$ for all $s \geq 0$



Hybrid Control Strategy – Switching Logic

Let $\dot{V}_1(z) := \langle \nabla V(z), f_P(z, \kappa_1(z)) \rangle$.

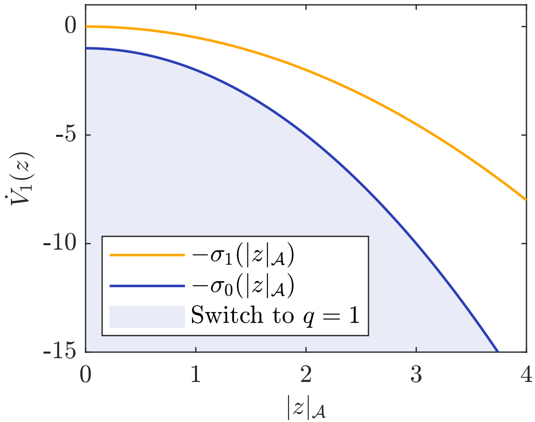
For $q = 0$:

$\dot{V}_1$ is "small enough to switch to $q = 1$" if

$$\dot{V}_1(z) \leq -\sigma_0(|z|_{\mathcal{A}})$$

and $\dot{V}_1$ is "large enough to hold $q = 0$" if

$$\dot{V}_1(z) \geq -\sigma_0(|z|_{\mathcal{A}}).$$



Hybrid Control Strategy – Switching Logic

Let $\dot{V}_1(z) := \langle \nabla V(z), f_P(z, \kappa_1(z)) \rangle$.

For $q = 1$:

$\dot{V}_1$ is "small enough to hold $q = 1$" if

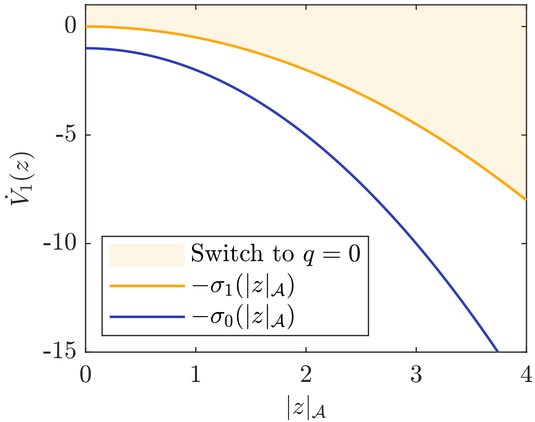
$$\dot{V}_1(z) \leq -\sigma_1(|z|_{\mathcal{A}})$$

and "large enough to switch to $q = 0$" if

$$\dot{V}_1(z) \geq -\sigma_1(|z|_{\mathcal{A}}).$$

A switch to $q = 0$ occurs only when both

$$\dot{V}_1(z) \geq -\sigma_1(|z|_{\mathcal{A}}) \quad \text{and} \quad V(z) \geq v.$$



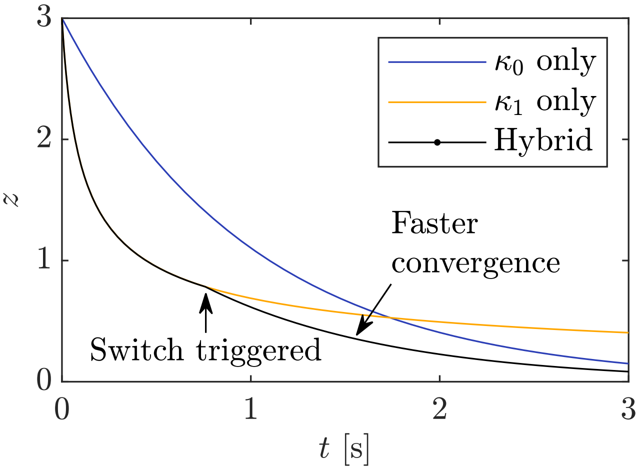
Example: Switching Logic

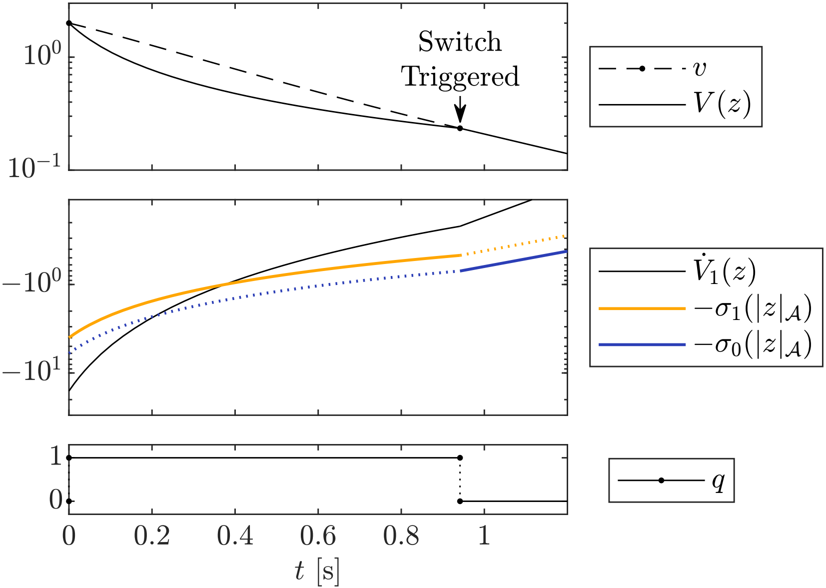
Consider the plant

$$\dot{z} = u$$

with $z, u \in \mathbb{R}$ and controllers

$$\kappa_0(z) := -z$$
$$\kappa_1(z) := -z^3$$





Switch
Triggered
10^0
10^{-1}
v
$V(z)$
-10^0
-10^1
$\dot{V}_1(z)$
$-\sigma_1(|z|_{\mathcal{A}})$
$-\sigma_0(|z|_{\mathcal{A}})$
1
0
q
0
0.2
0.4
0.6
0.8
1
t [s]

Dynamics of Closed-Loop System

At each switch:

- z is unchanged
- q is toggled to the opposite value in $\{0, 1\}$
- v is set to $V(z)$

Between switches:

- z evolves according to $\dot{z} = f_P(z, \kappa_q(z))$
- q is constant
- v evolves according to the dynamics chosen here:

$$\dot{v} = f_v(z, v, q) := \begin{cases} -v, & \text{if } q = 0, \\ -\sigma_1(|z|_{\mathcal{A}}) + \mu(V(z) - v), & \text{if } q = 1, \end{cases}$$

where $\mu > 0$ is parameter.

Hybrid Control Strategy — Design of v's Dynamics

$$\dot{v} = f_v(z, v, q) := \begin{cases} -v, & \text{if } q = 0, \\ -\sigma_1(|z|_{\mathcal{A}}) + \mu(V(z) - v), & \text{if } q = 1. \end{cases}$$

- With our switching logic, v converges to 0.
- After every switch from $q = 0$ to $q = 1$, there is an interval where no switches can occur because $V(z) < v$.
- While $q = 1$, if $V(z)$ does not converge fast enough then v will catch up to $V(z)$, causing a switch to $q = 0$.

Example: Linear Quadratic Regulator of Linearized System

Consider the nonlinear plant

$$\dot{z} = Az + u + \underbrace{f(z,u)}_{\text{Nonlinear component}} \qquad z, u \in \mathbb{R}^2.$$

Suppose the origin is GAS for

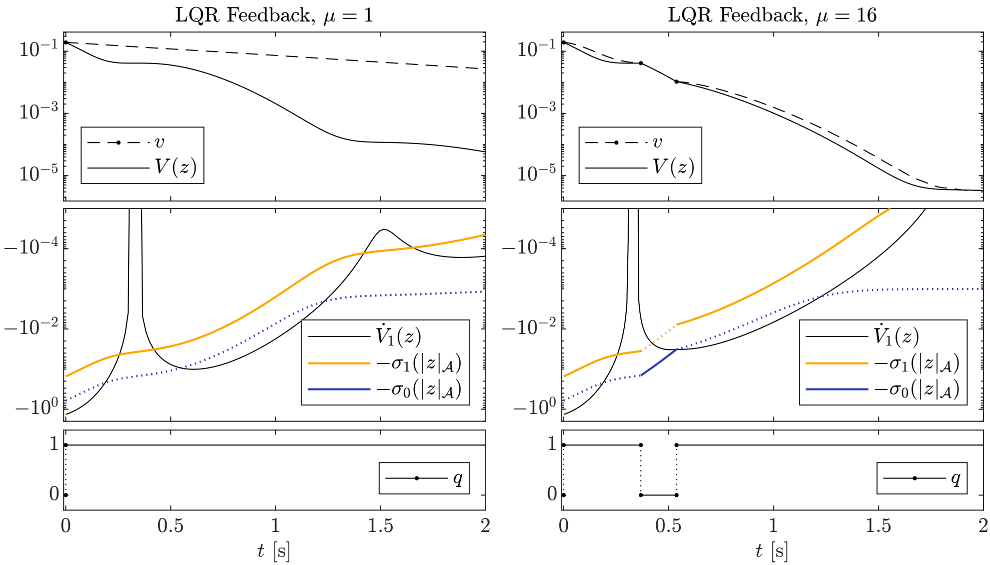
$$\kappa_0(z) := Kz.$$

For κ_1, we use the LQR feedback that solves the following LQR problem:

$$\begin{aligned} \underset{u}{\text{minimize}} \quad & \int_0^\infty \|z(t)\|^2 + \|u(t)\|^2 \, dt \\ \text{subject to} \quad & \dot{z} = Az + u. \end{aligned}$$

We could, instead, use MPC, machine learning, untested prototypes, etc.

Example: Linear Quadratic Regulator



Main Result: Global Asymptotic Stability

Theorem 1

Suppose that

- f_P, κ_0, *and* κ_1 *are continuous;*
- V *is continuously differentiable.*

Then, $\widetilde{\mathcal{A}} := \{(z, v, q) \mid z \in \mathcal{A}, v = 0\}$ *is GAS for the closed-loop system.*

Proof sketch. Let

$$\widetilde{V}(z, v, q) := \max\{V(z), v\}$$

We show that $\widetilde{V}$ is a Lyapunov function for the closed-loop system.

- Outside $\widetilde{\mathcal{A}}$, $\widetilde{V}(z, v, q)$ decreases along flows.
- At each switch, $\widetilde{V}(z, v, q)$ does not increase.
- The time between switches is nonzero.

Therefore, $\widetilde{\mathcal{A}}$ is GAS. □

Remark. The asymptotic stability of $\widetilde{\mathcal{A}}$ is robust to small perturbations.

Conclusion

Summary

- Lyapunov-certified controller acts as a backup to ensure convergence while using an uncertified controller.
- Exploit useful properties of an uncertified controller without losing the convergence guarantee.
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Future work

- Weaken assumptions on κ_1 and V.
- Consider systems with disturbances.
- Adapt hybrid control strategy for systems with constraints.

Questions?

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Definition of UGAS

Definition 1

A nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *uniformly globally stable* if there exists a continuous, strictly increasing function α such that every solution x to $\mathcal{H}$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \alpha(|x(0,0)|_{\mathcal{A}})$ for each $(t,j) \in \operatorname{dom} x$; and
- *uniformly globally attractive* for $\mathcal{H}$ if every maximal solution is complete and for all $\varepsilon > 0$ and $r > 0$, there exists $T > 0$ such that every solution x to $\mathcal{H}$ with $|x(0,0)|_{\mathcal{A}} \leq r$ satisfies $|x(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \operatorname{dom} x$ such that $t + j \geq T$.
- If $\mathcal{A}$ is both uniformly globally stable and uniformly globally attractive for $\mathcal{H}$, then it is said to be *uniformly globally asymptotically stable* (UGAS) for $\mathcal{H}$.

Because κ_0 is Lyapunov-certified there exists a Lyapunov function

$$V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$$

that guarantees $\mathcal{A}$ is UGAS for

$$\dot{z} = f_P(z, \kappa_0(z)).$$

Namely, there exist $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and a continuous positive definite function ρ such that

$$\begin{aligned} \alpha_1(|z|_{\mathcal{A}}) \leq V(z) &\leq \alpha_2(|z|_{\mathcal{A}}) & \forall z \in \mathbb{R}^n, \\ \dot{V}_0(z) &\leq -\rho(|z|_{\mathcal{A}}) & \forall z \in \mathbb{R}^n. \end{aligned}$$

Hybrid Systems

We consider hybrid systems modeled as

$$\mathcal{H}\begin{cases} \dot{x} = f(x) & x \in C \\ x^+ = g(x) & x \in D \end{cases}$$

with

- flow set $C \subset \mathbb{R}^n$
- flow map $f : C \to \mathbb{R}^n$
- jump set $D \subset \mathbb{R}^n$
- jump map $g : D \to \mathbb{R}^n$

Clarke Generalized Gradient

For the function

$$\widetilde{V}(x) := \max\{V(z), v\},$$

the Clarke generalized gradient at $x = (z, v, q)$ in the direction $w = (w_z, w_v, 0)$ is

$$\widetilde{V}^\circ(x, w) = \begin{cases} \langle \nabla_z V(z), w_z \rangle & \text{if } V(z) > v, \\ \max\{\langle \nabla_z V(z), w_z \rangle, w_v\} & \text{if } V(z) = v, \\ w_v & \text{if } V(z) < v. \end{cases} \qquad (1)$$