Conical Transition Graphs for Analysis of Asymptotic Stability in Hybrid Dynamical Systems

Paul K. Wintz[a,*], Ricardo G. Sanfelice[b]

[a] *University of California, Santa Cruz, Department of Applied Mathematics, 1156 High St., Santa Cruz, 95064, CA, USA*

[b] *University of California, Santa Cruz, Department of Electrical and Computer Engineering, 1156 High St., Santa Cruz, 95064, CA, USA*

Abstract

A method is proposed for analyzing asymptotic stability in so-called conical hybrid systems with modes, which include any hybrid systems where portion of the state excluding the logic variable is mapped linear at jumps and flows according to a constant or linear flow maps during flows, and where the flow and jump sets are conical. Specifically, this paper introduces the conical transition graph (CTG) to simplify the analysis of asymptotic stability in conical approximations by converting solutions to a hybrid system into walks through a discrete graph. By exploiting the fact that pre-asymptotic stability in a conical approximation implies pre-asymptotic stability in the original system, a CTG–based approach can establish asymptotic stability in hybrid systems that have nonlinear flow maps and jump maps without needing to construct a Lyapunov function. Discussion of how to reduce the size of the CTG allow for applying CTG analysis to systems where the CTG would have infinitely many vertices.

Keywords: Stability and stabilization of hybrid systems, Stability of nonlinear systems

1. Introduction

For continuous- and discrete-time systems, local asymptotic stability can be determined by linearizing the system and checking the eigenvalues of the resulting Jacobian matrix. For hybrid systems, however, the same ease is currently unavailable. In the conical approximation of a hybrid system, the flow and jumps sets are approximated by tangent cones, and the flow and jump maps are approximated by constant or linear approximations [1, Ch. 9]. It was shown in [2, Thm. 3.3] that the conical approximation of a hybrid system can be used to determine if a point is pre-asymptotically stable. Namely, if a point

*Corresponding Author
Email addresses: `pwintz@ucsc.edu` (Paul K. Wintz), `ricardo@ucsc.edu` (Ricardo G. Sanfelice)

is pre-asymptotically stable with respect to the conical approximation, then the center of the approximation in the original hybrid system is locally pre-asymptotically stable. (The prefix "pre-" indicates that some maximal solutions may terminate in finite time due to the solution leaving the region of the state space where it is permitted to evolve.) The utility of [2, Thm. 3.3] is currently limited, however, by the fact that it is still generally difficult to show that the origin of a conical approximation is pre-asymptotically stable. The purpose of this article is to close this gap by introducing the *conical transition graph* (CTG) as a tool to determine asymptotic stability in conical approximations. Thereby, we can establish local asymptotic stability in non-conical hybrid systems.

A graph-based approach is used in [3] to determine Lyapunov and asymptotic stability of a class of hybrid systems called piecewise constant derivatives (PCD). In a PCD system, the state space is partitioned into polyhedral regions with a flow vector field that is constant within each region but not necessarily continuous on their boundaries. The class of systems considered in the present work is more general in that the hybrid systems permit jumps in the value of the state and transitions between modes.

While there are limited results for analyzing stability of hybrid systems via conical approximations, there are numerous other approaches for stability analysis in the literature [4, 5] and [1, Thm. 7.30]. Lyapunov functions are a powerful and flexible tool for proving many types of stability properties, including stability of sets, finite-time stability, Zeno stability, and input to state stability [6, 7]. For hybrid systems where asymptotic stability of a limit cycle is of interest rather asymptotic stability of an equilibrium point, Poincaré maps have been used in hybrid systems to prove convergence of solutions to limit cycles [8, 9, 10]. Discrete graphs[1] have been used to evaluate stability of switched dynamical systems including discrete-time linear systems [11], discrete-time nonlinear systems [12], and continuous-time linear systems [13]. In contrast to the existing methods for switched systems, the present work is (to the best of our knowledge) the first graph-theoretic approach to analyze asymptotic stability in non-switched hybrid systems (i.e., systems where components of the state vector may range over a continuum at jumps). In the context of reachability analysis, [14] introduced *conical abstractions* as a graph-based method to compute infinite-horizon reachable sets for linear hybrid automata. The biggest drawback of the Lyapunov function method is that Lyapunov functions are often difficult to construct. There have, however, been advances made for algorithmically constructing Lyapunov functions. For hybrid systems defined by polynomial functions, Lyapunov functions can be constructed numerically via sum-of-squares (SOS) programming [15, 16, 17, 18, 19, 20]. Lyapunov functions can also be generated for non-polynomial systems by modeling non-polynomial functions as polynomials plus a disturbance, as done in [19] for barrier certificates, or by transforming the system into a polynomial system as done in [17] for continuous-time systems. The SOS approach to constructing Lyapunov functions is powerful but suffers from two limitations. Firstly, SOS requires solving a semidefinite program (SDP) that grows quickly as the dimension of the hybrid system and the degrees of the polynomials increase. While there are efficient algorithms for solving SDP's, the size of the optimization problem can make them

[1]Throughout, we use *graph* in the sense of *discrete graph*—that is, a set of vertices connected by edges or arrows.

computationally expensive for high-dimensional hybrid systems. Secondly, since SOS is a numerical approach, it requires the hybrid system to be fully defined, numerically—it cannot have any unspecified parameters. This inhibits using SOS to reason about parameters, limiting its utility for, e.g., designing an asymptotically stabilizing feedback law.

An alternative algorithmic approach to determine stability-like properties is via reachability analysis. The idea behind this approach is to use numerical reachability tools for hybrid systems [21, 14, 22, 23] to approximate the reachable set for solutions starting nearby an equilibrium and thereby assess stability numerically.

The conical transition graph is designed to simplify the analysis of asymptotic stability of isolated equilibria by creating a simplified representation of ways that solutions to a hybrid system can evolve continuously (called *flows*) or evolve discretely (called *jumps*). Collectively, we refer to flows and jumps as *transitions*. In particular, the CTG is a directed graph with set-valued weights assigned to each arrow. Each vertex in the CTG represents either the origin $0_n \in \mathbb{R}^n$ or a point in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, where each point $v \in \mathbb{S}^{n-1}$ acts as a representation of all the points in the ray $\{rv | r > 0\}$ spanned by v. In this way, we consider the projection of $\mathbb{R}^n$ onto $\mathbb{S}^{n-1} \cup \{0_n\}$, as shown in Figure 1. Roughly speaking, each arrow in the CTG represents the ways that solutions to a hybrid system, as projected onto $\mathbb{S}^{n-1} \cup \{0_n\}$, can transition (flow or jump) between points in $\mathbb{S}^{n-1} \cup \{0_n\}$. The weight of each arrow contains all possible relative changes in magnitude that a solution can exhibit when it undergoes the transition. Asymptotic stability can be determined from the products of walks through the CTG. Products converging to zero indicate convergence of solutions to the origin.

This article extends the author's previous work, [24], in two ways. First, this article defines and analyzes conical hybrid systems with modes—allowing switching between several regimes. To aid in analysis, we introduce in this article the concept of a CTG-simulation of a solution to a hybrid system. By showing a correspondence between solutions and CTG-simulations, we show that the CTG of a hybrid system can be used to determine asymptotic stability. Beyond the results in this article, CTG-simulations may be a useful theoretic tool in future work for using CTG's in reachability analysis.

Second, we describe how to reduce the size of a conical transition graph by an "abstraction" that groups together sets of vertices. By applying this method to conical transition graphs with large—possibly infinite—numbers of vertices, we can reduce intractable computational problems into problems that are solvable.

The remainder of this article is organized as follows. Preliminary concepts and notation are introduced in Section 2. In Section 2.2 we introduce conical hybrid systems with modes, and in Section 2.3 we describe the important radial homogeneity property of conical hybrid systems. We briefly describe two applications of conical hybrid systems in Section 3. Conical transition graphs are introduced in Section 4. Our results, in Section 5, demonstrate how to use a conical transition graph to determine pre-asymptotic stability in conical hybrid systems. Section 5.1 describes CTG-simulations, which is a useful tool in the subsequent theoretical developments. Our stability and pre-asymptotic stability results are found in Section 5.2. Section 6 describes our approach to reducing the size of

CTG's by creating an "abstract" CTG that groups together vertices.

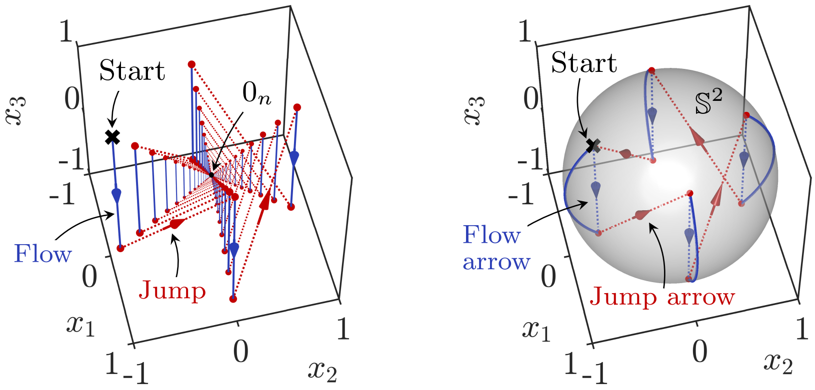

Figure 1: The evolution of solutions to a hybrid system on $\mathbb{R}^3$ (left) are reduced in the CTG (right) to discrete transitions on $\mathbb{S}^2$, which we label as *flow arrows* and *jump arrows*. In the right image, solid blue curves indicate continuous-time flows projected onto $\mathbb{S}^2$.

2. Preliminaries

For notation, we use $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. The Euclidean norm of $v \in \mathbb{R}^n$ is written $|v|$. We write the inner product between v_1 and v_2 in $\mathbb{R}^n$ as $\langle v_1, v_2 \rangle$. The zero vector in $\mathbb{R}^n$ is denoted 0_n. The domain of a function f is written $\operatorname{dom}(f)$. Given a set $S \subset \mathbb{R}^n$, we write the closure as $\overline{S}$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n : |x| = 1\}$, and the unit sphere plus the origin is written as

$$\mathbb{S}_0^{n-1} := \mathbb{S}^{n-1} \cup \{0_n\}. \tag{1}$$

The *normalized radial vector* function $\operatorname{nrv} : \mathbb{R}^n \to \mathbb{S}_0^{n-1}$ is defined for each $v \in \mathbb{R}^n$ as

$$\operatorname{nrv}(v) := \begin{cases} v/|v| & \text{if } v \neq 0_n \\ 0_n & \text{if } v = 0_n. \end{cases} \tag{2}$$

The following properties of the nrv function are used in this work.

$$\forall x \in \mathbb{R}^n : \quad x = |x| \operatorname{nrv}(x). \tag{3}$$

$$\forall x \in \mathbb{R}^n \text{ and } r > 0 : \quad \operatorname{nrv}(rx) = \operatorname{nrv}(x). \tag{4}$$

$$\forall x \in \mathbb{R}^n \text{ and } A \in \mathbb{R}^{n \times n} : \quad \operatorname{nrv}(Ax) = \operatorname{nrv}(\operatorname{nrv}(Ax)) = \operatorname{nrv}(A \operatorname{nrv}(x)). \tag{5}$$

Let $S \subset \mathbb{R}^n$ be nonempty and let $x \in \overline{S}$. The *contingent cone* $T_S(x)$ is the set of all vectors $v \in \mathbb{R}^n$ such that there exist a sequence of positive real numbers $h_i \to 0^+$ and a sequence of vectors $v_i \to v$ such that $x + h_i v_i \in S$ for all $i \in \mathbb{N}$ (see [25]). For any $S \subset \mathbb{R}^n$ and $x \in \overline{S}$, the contingent cone of S at x is a *cone*, meaning that for all $x \in T_S(x)$ and all $\alpha > 0$, we have that $\alpha x \in T_S(x)$.

For any $x \in \mathbb{R}^n$, we write the open ray from the origin through x as

$$\operatorname{ray}(x) := \{\alpha x \in \mathbb{R}^n \mid \alpha > 0\}$$

and the corresponding closed ray as

$$\overline{\operatorname{ray}}(x) := \{\alpha x \in \mathbb{R}^n \mid \alpha \geq 0\}.$$

Given a cone $K \subset \mathbb{R}$ and any $x \in \mathbb{R}^n$,

$$x \in K \iff \operatorname{ray} x \subset K.$$

We write the *conical hull* of $x_1, x_2, \ldots, x_p \in \mathbb{R}^n$ as

$$\operatorname{cone}(x_1, x_2, \ldots, x_p) = \{\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_p x_p \mid \alpha_i \geq 0\}.$$

Given a set $\mathcal{S} \subset \mathbb{R}^n$ and linear map $A \in \mathbb{R}^{n\times n}$, then the transformation of $\mathcal{S}$ by A is defined as

$$A\mathcal{S} := \{Ax \in \mathbb{R}^n \mid x \in \mathcal{S}\}.$$

2.1. Hybrid Systems

We consider hybrid systems on $\mathbb{R}^n$ in the form

$$\mathcal{H} : \begin{cases} \dot{x} \;\; = f(x) & x \in C, \\ x^+ \in G(x) & x \in D, \end{cases} \tag{6}$$

with state $x \in \mathbb{R}^n$, flow set $C \subset \mathbb{R}^n$, flow map $f : C \to \mathbb{R}^n$, jump set $D \subset \mathbb{R}^n$, and (set-valued) jump map $G : D \rightrightarrows \mathbb{R}^n$. The system $\mathcal{H}$ can be written compactly as $\mathcal{H} = (C, f, D, G)$. The continuous-time system formed by removing the discrete dynamics of $\mathcal{H}$ is written as (C, f).

A *hybrid time domain* E is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for every $(T, J) \in E$, there exists a sequence

$$0 = t_0 \leq t_1 \leq \cdots \leq t_{J+1} = T$$

such that

$$\begin{aligned} &E \cap ([0, T] \times \{0, 1, \ldots, J\}) \\ &\qquad = ([t_0, t_1]\times\{0\}) \cup ([t_1, t_2]\times\{1\}) \cup \cdots \cup ([t_J, t_{J+1}]\times\{J\}). \end{aligned} \tag{7}$$

Each t_1, t_2, $\ldots$, t_J in (7) is called a *jump time* in E. If $t_{j-1} < t_j$, then $[t_{j-1}, t_j]$ is called an *interval of flow* in E. A function $\phi : \operatorname{dom}(\phi) \to \mathbb{R}^n$ is called a *hybrid arc* if $\operatorname{dom}(\phi)$ is a hybrid time domain and ϕ is absolutely continuous on each interval of flow in $\operatorname{dom}(\phi)$. We write $\sup_t E := \sup\{t \in \mathbb{R}_{\geq 0} \mid (t, j) \in E\}$, $\sup_j E := \sup\{j \in \mathbb{N} \mid (t, j) \in E\}$, and $\operatorname{length}(E) := \sup_t E + \sup_j E$. If the domain E of a hybrid arc ϕ has length ∞, then ϕ is said to be *complete*. A hybrid arc ϕ is said be an *extension of* a hybrid arc ψ if $\operatorname{dom}(\psi)$ is a strict subset of $\operatorname{dom}(\phi)$ and

$$\phi(t, j) = \psi(t, j) \quad \forall (t, j) \in \operatorname{dom}(\psi).$$

Definition 1 (Hybrid Solution)**.** A hybrid arc ϕ is called a *solution* of $\mathcal{H}$ if it satisfies the following:

- At each jump time t_j in $\mathrm{dom}(\phi)$,

$$\phi(t_j, j-1) \in D \tag{8a}$$
$$\phi(t_j, j) \in G\big(\phi(t_j, j-1)\big). \tag{8b}$$

- For each interval of flow $[t_j, t_{j+1}]$ in $\mathrm{dom}(\phi)$ (for some j with t_{j+1} possibly infinite),

$$\phi(t,j) \in C \qquad \text{for all } t \in (t_j, t_{j+1}) \tag{9a}$$
$$\frac{d\phi}{dt}(t,j) \in F(\phi(t,j)) \qquad \text{for almost all } t \in (t_j, t_{j+1}). \tag{9b}$$

$\diamond$

A solution ϕ to $\mathcal{H}$ is said to be *maximal* if it cannot be extended, that is, if there does not exist another solution ψ to $\mathcal{H}$ such that $\mathrm{dom}(\phi)$ a strict subset of $\mathrm{dom}(\psi)$ (that is, $\mathrm{dom}\,\phi \subsetneq \mathrm{dom}\,\psi$) and $\phi(t,j) = \psi(t,j)$ for all $(t,j) \in \mathrm{dom}\,\phi$. At a jump time t in a hybrid domain, t corresponds to several values of j, so it is useful to define a function that maps each t to a single value of j. In particular, for each $(t,j) \in \mathrm{dom}\,\phi$, we define

$$t \mapsto \bar{\jmath}(t) := \max\{j \mid (t,j) \in \mathrm{dom}\,\phi\}.$$

For more background on hybrid systems, see [1, 26].

Given a nonempty set $\mathcal{A} \subset \mathbb{R}^n$, the distance from any $x \in \mathbb{R}^n$ to $\mathcal{A}$ is

$$|x|_{\mathcal{A}} := \inf_{a \in \mathcal{A}} |a - x|.$$

Definition 2. Consider a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$. A nonempty set $\mathcal{A} \subset \mathbb{R}^n$ is said to be

- *stable* for $\mathcal{H}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \delta$, we have that $|\phi(t,j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t,j) \in \mathrm{dom}\,\phi$,
- *pre-attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0,0)|_{\mathcal{A}} \leq \mu$, we have that $(t,j) \mapsto |\phi(t,j)|_{\mathcal{A}}$ is bounded and, if ϕ is complete, then

$$\lim_{t+j \to \infty} |\phi(t,j)|_{\mathcal{A}} = 0,$$

- *pre-asymptotically stable* (pAS) for $\mathcal{H}$ if $\mathcal{A}$ is stable and pre-attractive for $\mathcal{H}$,
- *asymptotically stable* for $\mathcal{H}$ if x_* is pAS for $\mathcal{H}$ and every maximal solution is complete. $\diamond$

Definition 3 (Forward Invariance)**.** A set $K \subset \mathbb{R}^n$ is said to be *forward pre-invariant* for a hybrid system $\mathcal{H}$ if, for each $x_0 \in K$ and each maximal solution ϕ starting from $\phi(0,0) = x_0$, then $\phi(t,j) \in K$ for all $(t,j) \in \mathrm{dom}\,\phi$. If, additionally, each maximal solution starting in K is complete, then K is called *forward invariant.* $\diamond$

Definition 4 (Conical Hybrid System with Modes)**.** Let $\mathcal{Q} := \{1, 2, \ldots, N_{\mathrm{Q}}\}$ be a finite set of modes, let $\mathcal{E} \subset \mathcal{Q} \times \mathcal{Q}$ be directed edges (transitions) between modes. Consider a hybrid system $\mathcal{H}$ with state $x := (q, z) \in \mathcal{Q} \times \mathbb{R}^n$ in the form

$$\begin{cases} \dot{x} = f(q,z) := \begin{bmatrix} 0 \\ f_q(z) \end{bmatrix} & x \in C := \left\{ \begin{bmatrix} q \\ z \end{bmatrix} \in \mathcal{Q} \times \mathbb{R}^n \,\middle|\, z \in C_q \right\} \\ x^+ \in G(q,z) := \left\{ \begin{bmatrix} q' \\ A_e z \end{bmatrix} \,\middle|\, \begin{array}{l} \exists e := (q,q') \in \mathcal{E} \\ \text{s.t. } z \in D_e \end{array} \right\} & x \in D := \left\{ \begin{bmatrix} q \\ z \end{bmatrix} \in \mathcal{Q} \times \mathbb{R}^n \,\middle|\, \begin{array}{l} \exists q' \in \mathcal{Q} \\ \text{s.t. } z \in D_{(q,q')} \end{array} \right\}, \end{cases} \tag{10}$$

where for each mode $q \in \mathcal{Q}$ and edge $e := (q, q') \in \mathcal{E}$, the function $z \mapsto f_q(z)$ is linear or constant, $A_e \in \mathbb{R}^{n \times n}$, the set $C_q \subset \mathbb{R}^n$ is a closed cone that defines the region where z is allowed to flow while in mode q, and for each $q' \in \mathcal{Q}$, the set $D_e \subset \mathbb{R}^n$ is a closed cone that defines the region where z is allowed to jump from mode q to mode q'. If $(q, q') \notin \mathcal{E}$, then $D_e = \varnothing$.

Since q does not depend on t, we write it as a function of j only when it occurs as a component of hybrid arcs, that is, $j \mapsto q(j)$. When mode q has linear flows, we write $\dot{z} = A_q z$, where $A_q \in \mathbb{R}^{n \times n}$, whereas when mode q has constant flows we write $\dot{z} = f_q^*$. $\diamond$

A diagram of a conical hybrid system with two modes is shown in Figure 2.

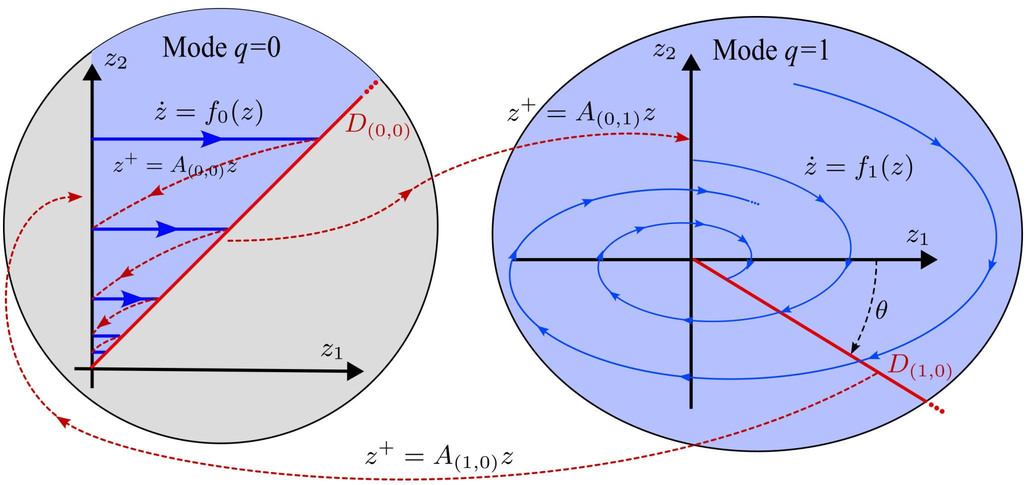

Figure 2: A conical hybrid system with two modes. Mode $q = 0$ (left) has constant flows and mode $q = 1$ (right) has linear flows.

Example 1 (Conical Hybrid System with Modes)**.** As an example of a hybrid system with modes, we consider a conical hybrid system $\mathcal{H}$ in $\mathbb{R}^2$ with two modes, $\mathcal{Q} := \{0, 1\}$, where mode $q = 0$ has constant flows and mode $q = 1$ has linear flow modes. For mode 0, let flows be defined by $\dot{z} = f_0^* := \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$ and

$$C_0 := \left\{(z_1, z_2) \in \mathbb{R}^2 \;\middle|\; z_2 \geq z_1,\ z_1 \geq 0\right\},$$

and let

$$D_{(0,0)} := D_{(0,1)} := \{(z_1, z_2) \mid z_1 = z_2,\ z_1 \geq 0\}.$$

After a jump from $q = 0$ to $q = 0$, the value of z is given by $z^+ = A_{(0,0)}z$, and after a jump from $q = 0$ to $q = 1$, it is given by $z^+ = A_{(0,1)}$, where

$$A_{(0,0)} := \begin{bmatrix} 0 & 0 \\ \lambda_0 & 0 \end{bmatrix} \quad \text{and} \quad A_{(0,1)} := \begin{bmatrix} 0 & 0 \\ \lambda_1 & 0 \end{bmatrix},$$

with $\lambda_0 > 0$, $\lambda_1 > 0$. Thus, at jumps, z is mapped to the z_2-axis.

For mode $q = 1$, let $\dot{z} = A_1 z$ where

$$A_1 := \begin{bmatrix} -2 & 4 \\ -2 & -1 \end{bmatrix},$$

and $C_1 := \mathbb{R}^2$. The jump set is defined as the ray from the origin with an angle $\theta(-\pi/2, \pi/2)$ from the z_1-axis, i.e., $D_{(1,0)} := \overline{\text{ray}}\left[\begin{smallmatrix} \cos\theta \\ \sin\theta \end{smallmatrix}\right]$. The jump map from $q = 1$ to $q = 0$ is defined by

$$A_{(1,0)} = \begin{bmatrix} \sin\theta & -\cos\theta \\ \cos\theta & \sin\theta \end{bmatrix},$$

which takes any vector $z \in D_{(1,0)}$ to $A_{(1,0)}z \in \{0\} \times \mathbb{R}_{\geq 0}$ (the z_2-axis). The transitions between modes are $\mathcal{E} = \{(0,0), (0,1), (1,0)\}$. Based on the choices of parameters λ_0, λ_1, and θ, the set $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ will be asymptotic stable or unstable. The techniques introduced in this article reduces the problem of checking stability into analyzing a discrete graph. ◇

2.3. Properties of Conical Hybrid Systems

An important property of conical hybrid systems, formalized in Proposition 1, below, is that their dynamics are radially homogenous—that is, a conical hybrid system behaves the same way at all distances from the origin, except for scaling effects.

Proposition 1. *Given a conical hybrid system with modes $\mathcal{H}$, let*

$$(t, j) \mapsto \phi(t, j) := \big(q(t, j), z(t, j)\big)$$

be a solution to $\mathcal{H}$. Then, for each $r > 0$, the hybrid arc $(t, j) \mapsto \psi_r(t, j)$ defined by

$$\psi_r(t, j) := \begin{bmatrix} q(j) \\ rz(\alpha_r(t), j) \end{bmatrix} \quad \forall(\alpha_r(t), j) \in \operatorname{dom}(\phi) \tag{11}$$

is also a solution to $\mathcal{H}$, where α_r is a class-$\mathcal{K}$ function defined, for all $(t, j) \in \operatorname{dom}(\phi)$, by

$$\alpha_r(t) = \int_0^t \delta_r(\bar{\jmath}(\tau))\, d\tau, \tag{12}$$

and

$$\delta_r(j) := \begin{cases} 1/r & \text{if } q(j) \text{ is a mode with constant flow} \\ 1 & \text{if } q(j) \text{ is a mode with linear flow.} \end{cases} \tag{13}$$

The effect of δ_r in (12) is that in modes with linear flow, the time ψ_r spends traversing an interval of flow matches ϕ, but in modes with constant flow, the time is dilated by a factor r.

Proof. First, we show that α_r is class-$\mathcal{K}$. From the definition, $\alpha_r(0) = 0$ and α_r is continuous. Since δ_r is strictly positive, α_r is monotonically increasing, so α_r is class-$\mathcal{K}$.

Let $J := \sup_j \operatorname{dom}(\phi)$, and let $t_1, t_2, \ldots, t_J$ be the jump times of ϕ. For ease of notation, let $t_0 := 0$ and, if J is finite, let $t_{J+1} := \sup_t \operatorname{dom}(\phi)$. For each jump time t_j in $\operatorname{dom}(\phi)$, the hybrid times $(t_j, j-1)$ and (t_j, j) are in $\operatorname{dom}(\phi)$, so $(\alpha_r^{-1}(t_j), j-1)$ and $(\alpha_r^{-1}(t_j), j)$ are in $\operatorname{dom}(\psi_r)$ (α_r is invertible because it is class-$\mathcal{K}$). Therefore, $t'_j := \alpha_r^{-1}(t_j)$ is a jump time in $\operatorname{dom}(\psi_r)$ for each $j \in \{1, 2, \ldots, J\}$.

Since α_r (and α_r^{-1}) is strictly increasing, $[t_j, t_{j+1}]$ is an interval of flow in $\operatorname{dom}(\phi)$ if and only if $[t'_j, t'_{j+1}]$ is an interval of flow in $\operatorname{dom}(\psi_r)$. For each $(t, j) \in \operatorname{dom}(\psi_r)$, let $(t, j) \mapsto z_r(t, j) := rz(\alpha_r(t), j)$, so that $\psi_r(t, j) = (q(j), z_r(t, j))$. Since $z(t_j, j-1) \in D_{(q(j-1),q(j))}$ and $D_{(q(j-1),q(j))}$ is a cone, we have that

$$z_r(t'_j,\, j-1) = rz\Big(\alpha_r\big(\alpha_r^{-1}(t_j)\big),\, j-1\Big) = rz(t_j,\, j-1) \in D_{(q(j-1),q(j))}.$$

Therefore, $\psi_r(t'_j, j-1) \in D$, so ψ_r satisfies (8a). Similarly, since $C_{q(j)}$ is a cone and $z(t, j) \in C_q$ for all $t \in (t_j, t_{j+1})$, we have that $z_r(t, j) \in C_q$ and thus $\psi_r(t, j) \in C$ for all $t \in (t'_j, t'_{j+1})$. Therefore, ψ_r satisfies (9a).

Now that we have established the jump times and intervals of flows of ψ_r, we want to show that ψ_r satisfies the jump and flow conditions in Equations (8) and (9). Take any $j \in \{1, 2, \ldots, J\}$. By (8b),

$$\phi(t_j, j) = \begin{bmatrix} q(j) \\ z(t_j, j) \end{bmatrix} \in G(\phi(t_j, j-1)),$$

so, from the definition of G in (10), $z(t_j, j) = A_{(q(j-1),q(j))} z(t_j, j-1)$. Thus, at $t'_j := \alpha_r^{-1}(t_j)$,

$$\psi_r(t'_j, j) = \begin{bmatrix} q(j) \\ rz\big(\alpha_r(\alpha_r^{-1}(t_j)), j\big) \end{bmatrix} = \begin{bmatrix} q(j) \\ A_{(q(j-1),q(j))}\big(rz(t_j, j-1)\big) \end{bmatrix}.$$

Since $D_{(q(j-1),q(j))}$ is a cone and $z(t_j, j-1)$ is in $D_{(q(j-1),q(j))}$, then $rz(t_j, j-1)$ is also in $D_{(q(j-1),q(j))}$. Therefore, $\psi_r(t_j, j)$ is in the set $G\big(\psi_r(t'_j, j-1)\big)$ as required by (8b).

If $t_{j+1} > t_j$, then $[t_j, t_{j+1}]$ is an interval of flow for ϕ, so for all $t \in (t_j, t_{j+1})$,

$$\dot{z}(t, j) = f_q(z(t, j)).$$

(Since f_q is linear or constant, we have that if $\dot{z} = f_q(z)$ for almost all $t \in (t_j, t_{j+1})$ then it, in fact, satisfies the ODE for all $t \in (t_j, t_{j+1})$). From Definition 4, the mode $q(j)$ has either linear or constant flows. Suppose, first, that $q(j)$ has linear flows. Then, for all $t \in (t_j, t_{j+1})$,

$$\frac{d}{dt}(z(t, j)) = A_{q(j)} z(t, j).$$

Applying the chain rule to $t \mapsto z_r(t,j) = rz(\alpha_r(t), j)$, we find

$$\dot{z}_r(t,j) = \frac{d}{dt}(rz(\alpha_r(t), j)) = A_{q(j)} rz(\alpha_r(t), j) \frac{d\alpha_r}{dt}(t) = A_{q(j)} z_r(t,j),$$

since $d\alpha_r/dt\,(t) = \delta_r(q(j)) = 1$ by applying the fundamental theorem of calculus to (13). Thus, ψ_r satisfies (9b) in the case of linear flows.

Suppose instead $q(j)$ has constant flows. Then, for all $t \in (t_j, t_{j+1})$,

$$\frac{d}{dt}(z(t_j, t_{j+1})) = f_{q(j)}.$$

Applying the chain rule to $t \mapsto z(\alpha_r(t), j)$, we find

$$\dot{z}_r(t,j) = \frac{d}{dt}(rz(\alpha_r(t), j)) = rf_{q(j)} \frac{d\alpha_r}{dt}(t) = \frac{rf_{q(j)}}{r} = f_{q(j)},$$

since $d\alpha_r/dt\,(t) = \delta_r(q(j)) = 1/r$. Thus, ψ_r satisfies (9b) in the case of linear flows. Therefore, ψ_r is a solution to $\mathcal{H}$. □

3. Applications of Conical Hybrid Systems

In this section, we introduce one application of conical hybrid systems.

3.1. Sampled Linear Systems

Example 2 (Linear System with Sampled Control)**.** Conical hybrid systems can be used to model and analyze linear control systems with sampled control updates. Consider the linear control system

$$\dot{z} = Az + Bu,$$

with state $z \in \mathbb{R}^n$ and input $u \in \mathbb{R}^m$. Suppose u is updated with period T according to $u := Kz$, where $K \in \mathbb{R}^{m \times n}$. One way to model such as system as a hybrid system is to use a timer variable $\tau \in [0, T]$ where $\dot{\tau} = 1$ and triggering events to update the input when $\tau = T$, and resetting $\tau^+ = 0$. Such an approach results in a non-conical hybrid system, because the set of τ-values where jumps are triggered is non-conical. As an alternative, we propose using a timer variable $\tau := (\tau_1, \tau_2) \in \mathbb{R}^2$ where τ evolves according to

$$\dot{\tau} = M\tau, \quad \text{with } M := \begin{bmatrix} 0 & -\omega \\ \omega & 0 \end{bmatrix},$$

where $\omega := \pi/T$. When τ starts with $\tau \in (0, \infty) \times \{0\}$, it takes time T for τ to reach $(-\infty, 0) \times \{0\}$. Thus, to achieve periodic sampling, we will update $u^+ = Kx$ and $\tau^+ = -\frac{1}{2}\tau$ whenever $\tau \in (-\infty, 0) \times \{0\}$. (The $\frac{1}{2}$ causes τ to converge to 0_n, which is convenient for showing the origin is asymptotically stable.) A representative trajectory for τ is shown in Figure 3.

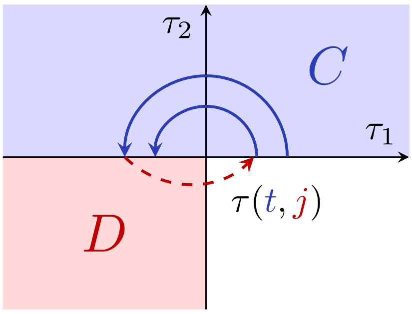

Figure 3: An example trajectory for the timer variable τ in Example 2.

The closed-loop system has state

$$x := (z, u, \tau) \in \mathcal{X} := \mathbb{R}^n \times \mathbb{R}^m \times \mathbb{R}^2,$$

and can be written as a conical hybrid system (without modes):

$$\begin{cases} \begin{bmatrix} \dot{z} \\ \dot{u} \\ \dot{\tau} \end{bmatrix} = \begin{bmatrix} Az + Bu \\ 0 \\ M\tau \end{bmatrix} & x \in C := \{(z, u, \tau_1, \tau_2) \in \mathcal{X} \mid \tau_2 \geq 0,\ \tau \neq (0, 0)\} \\ \begin{bmatrix} z^+ \\ u^+ \\ \tau^+ \end{bmatrix} = \begin{bmatrix} z \\ Ku \\ -\frac{1}{2}\tau \end{bmatrix} & x \in D := \{(z, u, \tau_1, \tau_2) \in \mathcal{X} \mid \tau_1 < 0,\ \tau_2 = 0\}. \end{cases}$$

Various adjustments to this example could allow for modeling systems that have nondeterministic delays between samples and switching between modes. ◇

3.2. Conical Approximations

One application of conical hybrid systems are as approximations of non-conical hybrid systems. Such approximations are called *conical approximations*. The following assumption is necessary for the conical approximation of a hybrid system $\mathcal{H}$ to be well-defined at a point $x_* \in \mathbb{R}^n$.

Assumption 1. For a given hybrid system $\mathcal{H} := (C, f, D, g)$ and $x_* \in \mathbb{R}^n$, suppose that the following conditions hold:

1. If $x_* \in \overline{D}$, then $g(x_*) = x_*$ and g is continuously differentiable at x_*.
2. If $x_* \in \overline{C}$, then f is continuous at x_*.
3. If $x_* \in \overline{C}$ and $f(x_*) = 0_n$, then f is continuously differentiable at x_*. ◇

Definition 5 ([1])**.** Given a hybrid system $\mathcal{H} = (C, f, D, g)$ and a point $x_* \in \mathbb{R}^n$ that satisfy Assumption 1, the *conical approximation* of $\mathcal{H}$ at x_* is

$$\breve{\mathcal{H}} : \begin{cases} \dot{x} &= \breve{f}(x) := \begin{cases} f(x_*), & \text{if } f(x_*) \neq 0 \\ A_{\mathrm{C}}(x - x_*), & \text{if } f(x_*) = 0, \end{cases} \qquad x \in \breve{C} := T_C(x_*), \\ x^+ &= \breve{g}(x) := A_{\mathrm{D}}(x - x_*), \qquad\qquad\qquad\qquad\quad x \in \breve{D} := T_D(x_*), \end{cases} \tag{14}$$

where A_{C} and A_{D} denote the Jacobian matrices of g and f at x_*, respectively:

$$A_{\mathrm{C}} := \frac{\partial f}{\partial x}(x_*) \quad \text{and} \quad A_{\mathrm{D}} := \frac{\partial g}{\partial x}(x_*). \qquad \diamond$$

The following result establishes local pre-asymptotic stability in a hybrid system via pre-asymptotic stability in its conical approximation.

Theorem 1 ([2], Thm. 3.3)**.** *Suppose a hybrid system $\mathcal{H}$ and a point $x_* \in \mathbb{R}^n$ satisfy Assumption 1. Let $\breve{\mathcal{H}}$ be the conical approximation of $\mathcal{H}$ at x_*. If 0_n is pAS for $\breve{\mathcal{H}}$, then x_* is locally pAS for $\mathcal{H}$.*

4. Conical Transition Graph

This work relies on definitions from graph theory, provided in this section. See [27] for details.

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ consists of a set of *vertices* $\mathcal{V}$ and a set of *arrows* $\mathcal{A}$. Each arrow in $\mathcal{G}$ starts at some vertex $v_1 \in \mathcal{V}$ and ends at some vertex $v_2 \in \mathcal{V}$. We write an arrow from v_1 to v_2 as $v_1 \to v_2$. In a directed graph, an arrow can have the same start and end point ($v_1 = v_2$), in which case it is called a *loop*.

We also allow for multiple arrows that have the same start and end points. To distinguish between such arrows, we assign each arrow a *label*. An arrow with the label "L" is written as $\mathfrak{a}^{\mathrm{L}} = v_1 \xrightarrow{\mathrm{L}} v_2$. In this work, we use only two labels: "F" and "J," which stand for "flow" and "jump." Thus, for $v_1, v_2 \in \mathcal{V}$, there can be at most two distinct arrows $v_1 \xrightarrow{\mathrm{F}} v_2$ and $v_1 \xrightarrow{\mathrm{J}} v_2$. If the label is irrelevant for a particular point of discussion, then it can be omitted.

A *weighted* directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$ is a directed graph $(\mathcal{V}, \mathcal{A})$ that also includes a weight function $\mathcal{W}$ that defines a weight for each arrow in $\mathcal{A}$. In a typical weighted graph, the weight function assigns a real number to each arrow, but in this work we use *set-valued* weights. Thus, the *weight function* is a set-valued map $\mathcal{W} : \mathcal{A} \rightrightarrows \mathbb{R}$ that maps each arrow $\mathfrak{a}$ in $\mathcal{A}$ to a set of real numbers $\mathcal{W}(\mathfrak{a}) \subset \mathbb{R}$.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$, a *walk* w through $\mathcal{G}$ is a finite or infinite sequence of arrows in $\mathcal{A}$. A walk of length $K \in \{1, 2, \dots\} \cup \{\infty\}$ is written

$$w = (\mathfrak{a}_0,\ \mathfrak{a}_1,\ \dots,\ \mathfrak{a}_{K-1}) = v_0 \to v_1 \to v_2 \to \cdots \to v_K,$$

such that $\mathfrak{a}_k = v_k \to v_{k+1}$ for each $k = 0, 1, \dots, K-1$.

We define the weight of a walk w as the cumulative *Minkowski set product* of the arrows in w. For any sets $A, B \subset \mathbb{R}$, the Minkowski set product of A and B is defined in [28] as $AB := \{ab \mid a \in A, b \in B\}$. For a finite-length walk $w = (\mathfrak{a}_0, \mathfrak{a}_2, \dots, \mathfrak{a}_{N-1})$, the *set-valued weight* of w is

$$\mathcal{W}(w) := \left\{ \prod_{k=0}^{N-1} r_k \;\middle|\; \begin{array}{l} r_k \in \mathcal{W}(\mathfrak{a}_k) \\ \quad \forall k = 0, 1, \dots, N-1 \end{array} \right\}. \tag{15}$$

If we let $K = \infty$, then $\mathcal{W}(w)$ may not be well-defined because the infinite product $\prod_{k=0}^{\infty} r_k$ in (15) may not converge. For this article, however, it is sufficient to define $\mathcal{W}(w)$ if and only if $\prod_{k=0}^{\infty} r_k$ converges to 0 for every choice of $\{r_k\}$. For an infinite-length walk $w := (\mathfrak{a}_0, \mathfrak{a}_1, \mathfrak{a}_2, \dots)$, we have that $\mathcal{W}(w) = \{0\}$ if and only if

$$\lim_{m \to \infty} \prod_{k=0}^{m} r_k = 0 \tag{16}$$

for every sequence $\{r_k\}_{k=0}^{\infty}$ with $r_k \in \mathcal{W}(\mathfrak{a}_k)$ for all $k \in \mathbb{N}$.

For an arrow $\mathfrak{a} \in \mathcal{A}$, we have that $\mathcal{W}(\mathfrak{a})$ is a set of real numbers, so we can write the *supremum weight* of $\mathfrak{a}$ as $\sup \mathcal{W}(\mathfrak{a})$. Similarly, for a walk w, we define $\sup \mathcal{W}(w)$ is the supremum weight of w.

Remark 1. Given a walk $w := (\mathfrak{a}_0, \mathfrak{a}_1, \dots, \mathfrak{a}_N)$ through a graph with set-valued weights, the supremum weight $\sup \mathcal{W}(w)$ is not always equal to the product of the supremum weights of the arrows. That is, in some cases

$$\sup \mathcal{W}(w) \neq \big(\sup \mathcal{W}(\mathfrak{a}_0)\big)\big(\sup \mathcal{W}(\mathfrak{a}_1)\big) \cdots \big(\sup \mathcal{W}(\mathfrak{a}_N)\big).$$

For example, if $\mathcal{W}(\mathfrak{a}_0) = \{0\}$ and $\mathcal{W}(\mathfrak{a}_1) = (1, \infty)$, then $\sup \mathcal{W}(w) = 0$ but the product $\big(\sup \mathcal{W}(\mathfrak{a}_0)\big)\big(\sup \mathcal{W}(\mathfrak{a}_1)\big) = 0 \cdot \infty$ is undefined. Thus, it is important that the supremum is evaluated *after* computing the product.

The CTG is designed to be a simplified representation of a conical hybrid system $\mathcal{H}$ to facilitate the analysis of pre-asymptotic stability. To this end, we exploit properties of conical hybrid systems, along with assumptions on the continuous dynamics of the hybrid system, so that the CTG can be used to establish that the origin of $\mathcal{H}$ is pAS. In particular, we exploit two simplifications.

In a conical hybrid system, Proposition 1 asserts that the distance a solution starts from the origin of does not affect the way it can evolve (aside from scaling effects). Thus, if we consider any ray from the origin and allow every point in the ray to evolve according to the dynamics of $\mathcal{H}$, then that ray is (in a sense) preserved. Using this observation, the first simplification in the CTG comes from using the nrv function to map $\mathbb{R}^n$ to $\mathbb{S}_0^{n-1}$ so that each single point $p \in \mathbb{S}_0^{n-1}$ represents every point in $\operatorname{ray}(p)$.

Mapping $\mathbb{R}^n$ to $\mathbb{S}_0^{n-1}$ reduces the dimension by one and—more importantly—allows for recurrent walks through the CTG despite convergence of solutions (see Figure 1). For example, suppose that for some $v \in \mathbb{S}^{n-1}$, a solution ϕ to $\mathcal{H}$ repeatedly enters $\operatorname{ray}(v)$. That is, $\phi(t_k, j_k) \in \operatorname{ray}(v)$ for a sequence of hybrid times $\{(t_k, j_k)\}$ in $\operatorname{dom}(\phi)$. Then,

$$v = \operatorname{nrv}(\phi(t_1, j_1)) = \operatorname{nrv}(\phi(t_2, j_2)) = \cdots .$$

Furthermore, the set of possible rays that ϕ can transition into from $\phi(t_k, j_k) \in \operatorname{ray}(v)$ via a single jump or flow is the same at every hybrid time (t_k, j_k) in the sequence. Exploiting this information allows us to uncover patterns in the behavior of $\mathcal{H}$.

By collapsing $\mathbb{R}^n$ to $\mathbb{S}_0^{n-1}$, however, we lose information about the magnitude (norm) of solutions. Instead, the weight of each arrow in the CTG typically contains every possible *relative* change of magnitude that a solution $(t, j) \mapsto \phi(t, j)$ can exhibit as $(t, j) \mapsto \operatorname{nrv}(\phi(t, j))$ moves from the arrow's start vertex to its end vertex (both in $\mathbb{S}_0^{n-1}$) via a single jump or a single interval of flow.

The second simplification arising from the CTG is that it allows us to partition the analysis of pre-asymptotic stability by considering separately solutions that are eventually continuous and solutions that are not eventually continuous. A hybrid arc is called *eventually continuous* if it has an interval of flow after the last jump time in its hybrid time domain. The aspects of eventually continuous solutions that are relevant to pre-asymptotic stability in $\mathcal{H} = (C, f, D, G)$ can be determined by analyzing the continuous-time system (C, f). In particular, our results assume that 0_n is pAS for (C, f)—which is necessary for 0_n to be pAS for $\mathcal{H}$ and can be verified using methods from continuous-time system analysis. Thus, the CTG is a tool for analyzing the behavior of solutions that are not eventually continuous.

Assuming that 0_n is pAS (and thus stable) for (C, f) has the added benefit that if we can show that a given solution converges to 0_n at jump times, then we can establish asymptotic convergence without analyzing the trajectories of solutions *during* intervals of flow. This is shown in the following lemma.

Lemma 1. *Let $\mathcal{H} = (C, f, D, G)$ be a conical hybrid system with modes. Suppose that $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ is stable for (C, f) and let ϕ be any solution to $\mathcal{H}$ with $\sup_j \operatorname{dom}(\phi) = \infty$. Then,*

$$\lim_{j\to\infty} |\phi(t_j, j)|_{\mathcal{O}} = 0 \implies \lim_{t+j\to\infty} |\phi(t, j)|_{\mathcal{O}} = 0,$$

where each t_j is the jth jump time in $\operatorname{dom}(\phi)$.

Proof. Let ϕ be any solution to $\mathcal{H}$ with $\sup_j \operatorname{dom}(\phi) = \infty$. Let t_1, t_2, ..., be the jump times of ϕ and suppose that

$$\lim_{j\to\infty} |\phi(t_j, j)|_{\mathcal{A}} = 0.$$

Take any $\varepsilon > 0$. We want to show that there exists $(t', j') \in \operatorname{dom}(\phi)$ such that $|\phi(t, j)|_{\mathcal{A}} < \varepsilon$ for all $(t, j) \in \operatorname{dom}(\phi)$ such that $t + j \geq t' + j'$.

For each j such that $[t_j, t_{j+1}]$ is an interval of flow in $\operatorname{dom}(\phi)$, the function $t \mapsto \phi(t, j)$ is a solution to (C, f) for all $t \in [t_j, t_{j+1}]$. By the stability of $\mathcal{O}$ for (C, f), there exists $\delta \in (0, \varepsilon)$ such that

$$|\phi(t_j, j)|_{\mathcal{A}} \leq \delta \implies |\phi(t, j)|_{\mathcal{A}} \leq \varepsilon \quad \forall t \in [t_j, t_{j+1}]. \tag{17}$$

Since $j \mapsto \phi(t_j, j)$ converges to $\mathcal{O}$, there exists $j' \in \mathbb{N}$ such that $|\phi(t_j, j)|_{\mathcal{A}} \leq \delta$ for all $j \geq j'$. Let $t' := t_{j'}$. Thus, from (17), we have that $|\phi(t, j)|_{\mathcal{A}} \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}(\phi)$ such that $t + j \geq t' + j'$. Since $\varepsilon > 0$ was arbitrary, we can take $\varepsilon \to 0$, thereby establishing that

$$\lim_{t+j\to\infty} |\phi(t,j)|_{\mathcal{A}} = 0. \qquad \square$$

As a consequence of Lemma 1, when determining whether persistently jumping solutions converge to $\mathcal{O}$ (e.g., to establish pre-asymptotic stability), we can ignore the interior of intervals of flow and only focus on showing that the solution at jump times converges. By doing so, we treat flows as discrete transitions that take solutions from their values immediately after a jump to their values immediately before the next jump. This effectively ignores the ordinary time required to traverse the flow because it is irrelevant for determining pre-asymptotic stability. Based on this fact, we generalize a flow that takes a solution ϕ from $x^{(0)} \in \mathbb{R}^n$ to $x^{(f)} \in \mathbb{R}^n$ in mode $q \in \mathcal{Q}$ as a flow arrow $(q, \mathrm{nrv}(x^{(0)})) \xrightarrow{\text{F}} (q, \mathrm{nrv}(x^{(f)}))$ in the CTG.

We design the CTG as a directed graph with set-valued weights with vertices that live in $\mathcal{Q} \times \mathbb{S}_0^{n-1}$. Each tuple $v := (q, s)$ in $\mathcal{Q} \times \mathbb{S}_0^{n-1}$ is a vertex in the CTG if it is possible for a solution to $\mathcal{H}$ to jump from or to v (i.e., if $v \in D \cup G(D)$). An arrow points between vertices $v_1 := (q_1, s_1)$ and $v_2 := (q_2, s_2)$ in the CTG if a solution to $\mathcal{H}$ can move directly from s_1 in mode q_1 to $\mathrm{ray}(v_2)$ in mode q_2 by a single jump or a single interval of flow. Each arrow is labeled by the type of transition it represents (either flow or jump). The weight of the arrow $v_1 \to v_2$ typically stores the relative change in the magnitude of a solution that starts at v_1 and ends in $\mathrm{ray}(v_2)$ (except if $s_1 = 0_n$, in which case the weight stores the absolute change—but the occurrence of such cases is limited). By multiplying together the weights of all the arrows in each walk through the CTG, we can analyze the relative change in distance of solutions from the origin (see Proposition 5, below).

Definition 6 (Conical Transition Graph)**.** Let $\mathcal{H} = (C, f, D, G)$ be a conical hybrid system on $\mathbb{R}^n$ with modes $\mathcal{Q}$. Let $\mathcal{L} := \{\text{“J”}, \text{“F”}\}$ be a set of labels (J stands for *jump* and F stands for *flow*). The CTG of $\mathcal{H}$ is a weighted, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$ where $\mathcal{V} \subset \mathcal{Q} \times \mathbb{S}_0^{n-1}$ is a set of *vertices*, $\mathcal{A} \subset \mathcal{V}^2 \times \mathcal{L}$ is a set of *arrows* between vertices, and $\mathcal{W} : \mathcal{A} \rightrightarrows \mathbb{R}_{\geq 0}$ is a set-valued *weight function* that assigns a set of nonnegative weights to each arrow. The set of vertices is defined as

$$\mathcal{V} := \big(D \cup G(D)\big) \cap \big(\mathcal{Q} \times \mathbb{S}_0^{n-1}\big). \tag{18}$$

For each $v^{\ominus} := (q^{\ominus}, s^{\ominus}) \in \mathcal{V} \cap D$, and each $(q^{\oplus}, z^{\oplus}) \in G(v^{\ominus})$, and for $s^{\oplus} := \mathrm{nrv}(z^{\oplus})$; a *jump arrow* $\mathfrak{a}^{\text{J}} = v^{\ominus} \xrightarrow{\text{J}} v^{\oplus}$ points from $v^{\ominus}$ to

$$v^{\oplus} := \big(q^{\oplus}, s^{\oplus}\big) = \big(q^{\oplus}, \mathrm{nrv}(A_e s^{\ominus})\big), \tag{19}$$

where $e := (q^{\ominus}, q^{\oplus})$. The weight of $\mathfrak{a}^{\text{J}} = v^{\ominus} \xrightarrow{\text{J}} v^{\oplus}$ is the (singleton) set

$$\mathcal{W}(\mathfrak{a}^{\text{J}}) := \big\{|z^{\oplus}|\big\} = \big\{|A_e s^{\ominus}|\big\}. \tag{20}$$

There is a *flow arrow* $\mathfrak{a}^{\text{F}} = v^{(0)} \xrightarrow{\text{F}} v^{(f)}$ from $v^{(0)} := (q, s^{(0)}) \in \mathcal{V} \cap G(D)$ to $v^{(f)} := (q, s^{(f)}) \in$

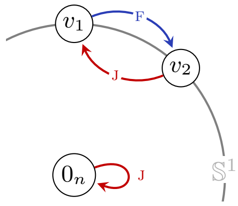

Figure 4: Conical transition graph for $\mathcal{H}$ in Example 3.

$\mathcal{V} \cap D$ if for some $\tau > 0$, there exists a function $\xi : [0, \tau] \to \mathbb{R}^n$ such that

$$\xi(0) = s^{(0)} \tag{21a}$$
$$\dot{\xi}(t) = f_q(\xi(t)) \quad \forall t \in (0, \tau) \tag{21b}$$
$$\xi(t) \in C_q \quad \forall t \in (0, \tau) \tag{21c}$$
$$\operatorname{nrv}(\xi(\tau)) = s^{(f)}. \tag{21d}$$

The weight of $\mathfrak{a}^{\mathrm{F}}$ is

$$\mathcal{W}(\mathfrak{a}^{\mathrm{F}}) := \big\{ |\xi(\tau)| \;\big|\; \xi : [0, \tau] \to \mathbb{R}^n \text{ satisfies (21) for some } \tau > 0 \big\}. \tag{22}$$

That is, for each solution $\xi : [0, \tau] \to \mathbb{R}^n$ of (21)—which has $|\xi(0)| = 1$ (if $|s^{(0)}| = 1$) or $|\xi(0)| = 0$ (if $|s^{(0)}| = 0$)—the magnitude of ξ at time τ is an element of the weight set: $|\xi(\tau)| \in \mathcal{W}(\mathfrak{a}^{\mathrm{F}})$. ◇

Note that each *vertex* in a CTG is a tuple containing a mode $q \in \mathcal{Q}$ and a vector $v \in \mathbb{R}^n$ with $v \in \mathbb{S}_0^{n-1}$.

If an arrow $\mathfrak{a} := v_1 \to v_2$ points from $v_1 := (q_1, s_1) \in \mathcal{V}$ to $v_2 := (q_2, s_2) \in \mathcal{V}$ with $s_1 \neq 0_n$, then the weight of $\mathfrak{a}$ is the set of all of the possible *relative* changes in the magnitude of a solution that transitions from $\operatorname{ray}(s_1)$ in mode q_1 to $\operatorname{ray}(s_2)$ in mode q_2 via a single jump or interval of flow (the mode can change only for jump arrows. For a flow arrow, $q_1 = q_2$). On the other hand, if $s_1 = 0_n$, then the weight of $\mathfrak{a}$ is the set of all of the possible *absolute* changes in magnitude for a transition from 0_n to $\operatorname{ray}(s_2)$ via a single jump or interval of flow (the relative change of distance is undefined because the initial distance 0 would be in the denominator).

In the following example, we consider a conical hybrid system with a single mode, so we omit the logic variable. In particular, we will consider only mode $q = 0$ from Example 1. To simplify the exposition, we will omit the mode variable "q" during this example.

Example 3. Consider the following conical hybrid system on $\mathbb{R}^2_{\geq 0}$ (the non-negative quadrant of $\mathbb{R}^2$) with a single mode:

$$\mathcal{H}\colon \begin{cases} f(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \forall x \in C := \big\{ x \in \mathbb{R}^2_{\geq 0} \;\big|\; x_2 \geq x_1 \big\}, \\ G(x) := \begin{bmatrix} 0 \\ \gamma x_1 \end{bmatrix} & \forall x \in D := \overline{\operatorname{ray}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] = \big\{ x \in \mathbb{R}^2_{\geq 0} \;\big|\; x_2 = x_1 \big\}, \end{cases} \tag{23}$$

with $\gamma > 0$. We will construct the CTG for $\mathcal{H}$. Let $v_1 := \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]$ and $v_2 := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, so $G(D) = \overline{\operatorname{ray}}\, v_1$ and $D = \overline{\operatorname{ray}}\, v_2$. Thus, the set of vertices is

$$\mathcal{V} = \big(\{0_n\} \cup \operatorname{ray} v_1 \cup \operatorname{ray} v_2\big) \cap \mathbb{S}_0^{n-1} = \{0_n, v_1, v_2\}$$

and the set of arrows is

$$\mathcal{A} = \{\underbrace{0_n \xrightarrow{\text{J}} 0_n,\ v_2 \xrightarrow{\text{J}} v_1}_{\text{Jump arrows}},\ \underbrace{v_1 \xrightarrow{\text{F}} v_2}_{\text{Flow arrow}}\}.$$

The CTG of $\mathcal{H}$ is depicted in Figure 4. ◇

Example 4 (Example 1, cont.)**.** Now, we will consider the full conical hybrid system $\mathcal{H}$ with modes from Example 1. By examining Figure 2 and the data of the system, we find that the vertices in the CTG are

$$\big(0, 0_n\big), v_0 := \big(0, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\big), v_1 := \big(0, \operatorname{nrv}\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]\big), \big(1, 0_n\big), v_2 := \big(1, \left[\begin{smallmatrix}0\\1\end{smallmatrix}\right]\big), v_3 := \big(1, \left[\begin{smallmatrix}\cos\theta\\ \sin\theta\end{smallmatrix}\right]\big),$$

and the arrows are

$$\begin{array}{lll} \big(0, 0_n\big) \xrightarrow{\text{J}} \big(0, 0_n\big), & \big(1, 0_n\big) \xrightarrow{\text{J}} \big(1, 0_n\big), & \big(1, 0_n\big) \xrightarrow{\text{F}} \big(1, 0_n\big), \\ \big(0, 0_n\big) \xrightarrow{\text{J}} \big(1, 0_n\big), & \big(1, 0_n\big) \xrightarrow{\text{J}} \big(0, 0_n\big), & v_0 \xrightarrow{\text{F}} v_1, \\ v_1 \xrightarrow{\text{J}} v_0, & v_1 \xrightarrow{\text{J}} v_2, & v_2 \xrightarrow{\text{F}} v_3, \\ v_3 \xrightarrow{\text{J}} v_0. & & \end{array} \tag{24}$$

There is not a flow arrow from v_3 to v_2 because flow arrows must start in $G(D)$, nor is $v_2 \xrightarrow{\text{F}} v_2$ because flow arrows must end in D. ◇

The need for the weights to be set-valued comes from the fact that there may be multiple solutions to (21) with different final magnitudes, $|\xi(T)|$, as in (22). The following example presents a conical hybrid system with a flow arrow that has a non-singleton weight.

Example 5. Consider the following conical hybrid system:

$$\mathcal{H} : \begin{cases} \dot{x} \;\; = f(x) := -1 & x \in C := \mathbb{R}_{\geq 0}, \\ x^+ = G(x) := x/2 & x \in D := \mathbb{R}_{\geq 0}. \end{cases}$$

Every maximal solution to $\mathcal{H}$ evolves by a non-deterministic combination of flows and jumps until it reaches 0_n, at which point it must jump from 0_n to 0_n forevermore. Thus, 0_n is pre-asymptotically stable for $\mathcal{H}$.

The vertex set of the CTG is $\mathcal{V} = \{0, 1\}$ and the arrow set is

$$\mathcal{A} = \{0 \xrightarrow{\text{J}} 0,\ 1 \xrightarrow{\text{J}} 1,\ 1 \xrightarrow{\text{F}} 0,\ 1 \xrightarrow{\text{F}} 1\}.$$

Consider, in particular, the arrow $1 \xrightarrow{\text{F}} 1$. For all $T \in (0,1)$, the function

$$\begin{aligned} \xi &: [0,T] \to \mathbb{R}_{\geq 0} \\ t &\mapsto \xi(t) := 1 - t \end{aligned}$$

satisfies (21) with $v^{(0)} := 1$, $v^{(\text{f})} := 1$, and

$$|\xi(T)| = 1 - T \in (0,1).$$

Thus, $1 \xrightarrow{\text{F}} 1$ is a flow arrow in the CTG with set-valued weight $\mathcal{W}(1 \xrightarrow{\text{F}} 1) = (0,1)$. $\diamond$

Whereas non-singleton weights for conical hybrid systems with constant flows are typically continuous intervals, such as $(0,1)$, for conical hybrid systems with linear flows, non-singleton weights are infinite sets of discrete points, as shown in the next example.

Example 6. Consider the conical hybrid $\mathcal{H} = (C, f, D, G)$ on $\mathbb{R}^2$ with dynamics given by

$$\begin{cases} f(x) := Ax & \forall x \in C := \mathbb{R}^2 \\ G(x) := \begin{bmatrix} -x_1 \\ 0 \end{bmatrix} & \forall x \in D := \overline{\operatorname{ray}} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \end{cases}$$

where $A := \left[\begin{smallmatrix} \gamma & -1 \\ 1 & \gamma \end{smallmatrix}\right]$ and $\gamma \in \mathbb{R}$. The vertex set for the conical transition graph is $\mathcal{V} = \{0_n, \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]\}$. It can be shown that there are two jump arrows $0_n \xrightarrow{\text{J}} 0_n$ and $\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \xrightarrow{\text{J}} \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$, and one flow arrow $\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right] \xrightarrow{\text{F}} \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right]$ (recall that the start of a flow arrow must be in $G(D)$ and the end must be in D). The weights for the jump arrows are

$$\mathcal{W}(0_n \xrightarrow{\text{J}} 0_n) = \{0\} \quad \text{and} \quad \mathcal{W}(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \xrightarrow{\text{J}} \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]) = \{1\}.$$

Solutions to (21) for the flow arrow $\mathfrak{a}^{\text{F}} := \left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right] \xrightarrow{\text{F}} \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]$ are given for each $T \in \{\pi, 3\pi, 5\pi, \dots\}$ by

$$t \mapsto \xi(t) := \exp(\gamma t) \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \quad \forall t \in [0,T].$$

At $t = T$, the magnitude of ξ is $|\xi(T)| = \exp(\gamma T)$. Thus, the weight of $\mathfrak{a}$ is

$$\mathcal{W}(\mathfrak{a}^{\text{F}}) = \{\exp(\gamma T) \mid T = \pi, 3\pi, 5\pi, \dots\}.$$

$\diamond$

In addition to having a non-singleton weight, the flow arrow $1 \xrightarrow{\text{F}} 1$ in Example 5 illustrates an exceptional case that we must consider. In Example 5, the origin is pAS for $\mathcal{H}$, so we want every infinite-length walk through the CTG to have weight $\{0\}$ (see Theorem 2, below). But, the weight of $w := 1 \xrightarrow{\text{F}} 1 \xrightarrow{\text{F}} 1 \xrightarrow{\text{F}} \cdots$ is actually $\mathcal{W}(w) = [0,1)$. To see $\mathcal{W}(w)$ contains $(0,1)$, take any $s > 0$ and let $r_k := \exp(-s/2^{k+1})$, which is in $\mathcal{W}(1 \xrightarrow{\text{F}} 1) = (0,1)$ for each $k \in \mathbb{N}$. Then, by selecting $\{r_k\}_{k=0}^{\infty}$ in (15), we compute

$$\prod_{k=0}^{\infty} r_k = \exp\bigl(-s/2 - s/4 - s/8 - \cdots\bigr) = e^{-s} \in (0,1).$$

Alternatively, selecting $r_k := 1/2 \in \mathcal{W}(1 \xrightarrow{\text{F}} 1)$ results in $\prod_{k=0}^{\infty} 1/2 = 0$. Hence, $\mathcal{W}(w) = [0, 1)$. The crux of the problem is that by repeatedly traversing the loop $1 \xrightarrow{\text{F}} 1$, the walk w represents a solution that flows part of the way to the origin, then flows a little more, and a little more, *ad infinitum*, without ever jumping. As indicated by the weight $\mathcal{W}(w)$, we can construct such as sequence of flows that will converge to 0, but also sequences that converge to any value in $[0, 1)$. Fortunately, any finite sequence of consecutive flow arrows can be replaced by a single flow arrow, whereas any infinite sequence of flow arrows represents a solution that never jumps, so we analyze it using continuous-time methods instead of the CTG. Therefore, we exclude walks with consecutive flow arrows from consideration.

Definition 7 (Well-formed Walk)**.** We say that a walk w through a conical transition graph $\mathcal{G}$ is *well-formed* if no pair of consecutive arrows in w are both flow arrows. That is, $w = (\mathfrak{a}_0, \mathfrak{a}_1, \ldots, \mathfrak{a}_{N-1})$ is a well-formed walk through $\mathcal{G}$ if for every $i \in \{1, 2, \ldots, N-1\}$, either $\mathfrak{a}_{i-1}$ or $\mathfrak{a}_i$ is a jump arrow. $\diamond$

Remark 2. A well-formed walk may include consecutive jump arrows.

5. Establishing Pre-asymptotic Stability via the CTG

This section presents a result that allows for pre-asymptotic stability of $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ (the combined origins of all of the modes) to be established by analyzing the CTG. For $\mathcal{O}$ to be pre-asymptotically stable, $\mathcal{O}$ must be forward invariant. Forward invariance of $\mathcal{O}$ can be easily checked for a conical hybrid system, as asserted by the following result.

Proposition 2. *The set $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ is not forward invariant with respect to conical hybrid system $\mathcal{H}$ if and only if it has a mode $q_{\text{c}} \in \mathcal{Q}$ with constant flows and $f_{q_{\text{c}}} \in C_{q_{\text{c}}} \setminus \{0_n\}$. Furthermore, if $\mathcal{O}$ is not forward invariant, then there exists a complete solution ϕ to $\mathcal{H}$ such that*

$$\lim_{t+j\to\infty} |\phi(t, j)|_{\mathcal{O}} = \infty.$$

Proof. Suppose $\mathcal{O}$ is not forward invariant. From the definition of G in (10), we find $G(\mathcal{O}) \subset \mathcal{O}$, so solutions to $\mathcal{H}$ cannot leave the origin at jumps. Thus, for some $q_{\text{c}} \in \mathcal{Q}$, solutions to $\mathcal{H}$ can flow away from the origin. Flows in q_{c} are either linear or constant. In both cases, the flow map is Lipschitz continuous, so solutions are unique. If flows are linear, then $f_{q_{\text{c}}}(0_n) = 0_n$, so all solutions to that start in $\{q_{\text{c}}\} \times \{0_n\}$ remain in $\{q_{\text{c}}\} \times \{0_n\}$. Hence, flows cannot be linear. Similarly, if flows are constant and $f_{q_{\text{c}}} = 0_n$, then solutions cannot leave the origin, so we must have constant flows with $f_{q_{\text{c}}} \neq 0_n$.

It remains to be shown that $f_{q_{\text{c}}} \in C$. If $f_{q_{\text{c}}} \notin C$, then any solution to $\dot{z} = f_{q_{\text{c}}}$ from $z(0) = 0_n$ immediately leaves $C_{q_{\text{c}}}$, so solutions to $\mathcal{H}$ cannot flow from the origin, contradicting our assumption that the origin is not forward invariant. Therefore, $f_{q_{\text{c}}} \in C_{q_{\text{c}}} \setminus \{0_n\}$.

To prove the converse direction, suppose mode q_{c} has constant flows and $f_{q_{\text{c}}} \in C_{q_{\text{c}}} \setminus \{0_n\}$. Then $\phi : \mathbb{R}_{\geq 0} \times \{0\} \to \mathbb{R}^n$ defined by

$$\phi(t, 0) := (q_{\text{c}}, t f_{q_{\text{c}}}) \quad \forall t \geq 0$$

is a complete solution to $\mathcal{H}$. (Since $f_{q_c} \in C_{q_c}$ and C_{q_c} is a cone, tf_{q_c} is also in C_{q_c} for all $t \geq 0$.) Finally, since $f_{q_c} \geq 0$, we have that $|\phi(t,j)|_{\mathcal{O}} \to \infty$. □

For a simple illustration of Proposition 2, consider $\mathcal{H}$ on $\mathbb{R}^n$ with a single mode q that has constant flows $\dot{z} = f_q^*$ and a flow set consisting of a single ray, $C := \overline{\operatorname{ray}}\, f_q^*$, where $f_q^* \in \mathbb{S}^{n-1}$. We have $f_q^* \in C$, so, by Proposition 2, the set $\mathcal{O}$ is not forward invariant for $\mathcal{H}$. The hybrid arc $\phi : \mathbb{R}_{\geq 0} \times \{0\} \to \{q\} \times \mathbb{R}^n$ defined by $\phi(t,j) := \big(q, tf_q^*\big)$ for all $(t,j) \in \operatorname{dom}(\phi)$ is a complete solution to $\mathcal{H}$ that leaves 0_n, and $\lim_{t+j\to\infty}|\phi(t,j)|_{\mathcal{O}} = \infty$.

5.1. CTG Simulations

This section establishes a correspondence between solutions to a conical hybrid system and walks through the CTG. For each solution, there is a unique walk called the CTG-simulation of that solution (Definition 8). That a CTG-simulation is, in fact, a walk through the CTG is asserted in Proposition 3. Conversely, Proposition 4 asserts that for every well-formed nonempty walk through the CTG of a hybrid system that starts and ends with a jump arrow, there exists a solution that the walk simulates. This section is concluded with Proposition 5, which shows that the relative change in the magnitude of a solution is an element in the set-valued weight of the solutions CTG-simulation.

Definition 8 (CTG Simulation)**.** Let $\mathcal{H}$ be a conical hybrid system with modes $\mathcal{Q}$ and conical transition graph $\mathcal{G}$. Consider any solution $(t,j) \mapsto \phi(t,j) := \big(q(j), z(t,j)\big)$ to $\mathcal{H}$ that jumps at least once. Let $J := \sup_j \operatorname{dom}(\phi) \in \{1, 2, \ldots, \infty\}$. Let $t_0 := 0$ and let t_j denote the jth jump time of ϕ for each $j \in \{1, 2, \ldots, J\}$. Let $K_0 := 0$ and for each finite $j \in \{1, \ldots, J\}$, let K_j be the cumulative number of jumps and intervals of flow in ϕ between $(t_1, 0) \in \operatorname{dom}(\phi)$ and $(t_j, j) \in \operatorname{dom}(\phi)$. Let $h_0 := (t_1, 0)$, and for each $k \in \{1, \ldots, K_J\}$, let h_k be the first hybrid time among

$$(t_1, 1),\ (t_2, 1),\ (t_2, 2),\ \ldots,\ (t_J, J-1),\ (t_J, J) \tag{25}$$

that does not occur among h_0, h_1, …, h_{k-1}. We denote the t-component of h_k as $\pi_{\mathrm{T}}(h_k)$ and the j-component as $\pi_{\mathrm{J}}(h_k)$, i.e., $h_k = (\pi_{\mathrm{T}}(h_k), \pi_{\mathrm{J}}(h_k))$. Note that for each $k \in \{0, 1, \ldots, K_J - 1\}$, either $\pi_{\mathrm{J}}(h_k) = \pi_{\mathrm{J}}(h_{k+1})$ and $\pi_{\mathrm{T}}(h_k) < \pi_{\mathrm{T}}(h_{k+1})$, or $\pi_{\mathrm{J}}(h_k) < \pi_{\mathrm{J}}(h_{k+1})$ and $\pi_{\mathrm{T}}(h_k) = \pi_{\mathrm{T}}(h_{k+1})$.

We say that

$$w := (v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \cdots \xrightarrow{\ell_{(K_J-2)}} v_{(K_J-1)} \xrightarrow{\ell_{(K_J-1)}} v_{K_J})$$

is the *CTG simulation* or the *$\mathcal{G}$-simulation* of ϕ, where $\{v_k\}_{k=0}^{K_J}$ is a sequence in $\mathcal{Q} \times \mathbb{S}_0^{n-1}$ defined as

$$v_k := \big(q(h_k), \operatorname{nrv}(z(h_k))\big) \quad \forall k \in \{0, 1, \ldots, K_J\} \tag{26}$$

and $\{\ell_k\}_{k=0}^{K_J-1}$ is a sequence of labels in $\mathcal{L}$ defined by

$$\ell_k := \begin{cases} \mathrm{J} & \text{if } \pi_{\mathrm{J}}(h_{k+1}) > \pi_{\mathrm{J}}(h_k) \\ \mathrm{F} & \text{if } \pi_{\mathrm{T}}(h_{k+1}) > \pi_{\mathrm{T}}(h_k) \end{cases} \quad \forall k \in \{0, 1, \ldots, K_J - 1\}. \tag{27}$$

◇

Remark 3. A CTG simulation of a solution ϕ is a representation of ϕ with "snapshots" of the solution projected onto $\mathbb{S}_0^{n-1}$ by the nrv function before and after each jump. Such a simulation does not say anything about how ϕ flows before the first jump or after the last jump.

Lemma 2. *Suppose $\mathcal{H} := (C, f, D, G)$ is a conical hybrid system with modes $\mathcal{Q}$ and transitions $\mathcal{E}$. For any $(q^{\ominus}, z^{\ominus}) \in D$ and $(q^{\oplus}, z^{\oplus}) \in G(q^{\ominus}, z^{\ominus})$, let $s^{\ominus} := \mathrm{nrv}(z^{\ominus})$ and $s^{\oplus} := \mathrm{nrv}(z^{\oplus})$. Then,*

$$v^{\ominus} := \big(q^{\ominus}, s^{\ominus}\big) \in \mathcal{V} \cap D, \quad v^{\oplus} := \big(q^{\oplus}, s^{\oplus}\big) \in \mathcal{V} \cap G(D),$$

and $\mathfrak{a}^{\mathrm{J}} := v^{\ominus} \xrightarrow{\mathrm{J}} v^{\oplus}$ is a jump arrow in the CTG of $\mathcal{H}$.

Furthermore, if $(t, j) \mapsto \phi(t, j) = \big(q(j), z(t, j)\big)$ is a solution to $\mathcal{H}$, then for each jump time t_j in $\mathrm{dom}(\phi)$,

$$\big(q(j-1), \mathrm{nrv}(z(t_j, j-1))\big) \xrightarrow{\mathrm{J}} \big(q(j), \mathrm{nrv}(z(t_j, j))\big) \tag{28}$$

is a jump arrow in $\mathcal{G}$.

Proof. Take any $(q^{\ominus}, z^{\ominus}) \in D$ and $(q^{\oplus}, z^{\oplus}) \in G(q^{\ominus}, z^{\ominus})$. It follows immediately from the definition of the jump set that $e := (q^{\ominus}, q^{\oplus}) \in \mathcal{E}$ and $z^{\ominus} \in D_e$. Since D_e is a cone, $s^{\ominus} := \mathrm{nrv}(z^{\ominus})$ is also in D_e, so $v^{\ominus} := (q^{\ominus}, s^{\ominus}) \in \mathcal{V} \cap D$.

Next, we will show that $v^{\oplus} := (q^{\oplus}, s^{\oplus})$ is a vertex in $\mathcal{V} \cap G(D)$ (specifically, $v^{\oplus} \in \mathcal{V} \cap G(D)$), where $s^{\oplus} := \mathrm{nrv}(z^{\oplus})$. Let

$$z^* := \begin{cases} s^{\ominus}/|A_e s^{\ominus}| & \text{if } A_e s^{\ominus} \neq 0_n \\ s^{\ominus} & \text{if } A_e s^{\ominus} = 0_n. \end{cases}$$

Since $s^{\ominus} \in D_e$ and D_e is a cone, we have that $z^* \in D_e$, so $(q^{\ominus}, z^*) \in D$. Then,

$$A_e z^* = s^{\oplus}.$$

To see why, first suppose that $A_e s^{\ominus} \neq 0_n$. Then,

$$A_e z^* = A_e(s^{\ominus}/|A_e s^{\ominus}|) = \mathrm{nrv}(A_e s^{\ominus}) = \mathrm{nrv}(A_e z^{\ominus}) = s^{\oplus},$$

where the penultimate equality is a result of (5). On the other hand, if $A_e s^{\ominus} = 0_n$, then $A_e z^* = 0_n = s^{\oplus}$. Therefore, $v^{\oplus} \in G(q^{\ominus}, z^*)$, so $v^{\oplus}$ is in $G(D)$ and $\mathcal{V}$.

To finish the proof, we must show that $v^{\ominus} \xrightarrow{\mathrm{J}} v^{\oplus}$ is a jump arrow in the CTG of $\mathcal{H}$. Using the definitions of $s^{\oplus}$ and $z^{\oplus}$, we have that $s^{\oplus} = \mathrm{nrv}(z^{\oplus}) = \mathrm{nrv}(A_e z^{\ominus})$. By linearity, $A_e z^{\ominus} = |z^{\ominus}| A_e s^{\ominus}$, so $\mathrm{nrv}(A_e z^{\ominus}) = \mathrm{nrv}(A_e s^{\ominus})$. Therefore, per (19), $v^{\ominus} \to v^{\oplus}$ is a jump arrow.

Finally, (28) is a jump arrow in $\mathcal{G}$ since $\phi(t_j, j-1) \in D$ at each jump time t_j. □

Lemma 3. *Consider a conical hybrid system $\mathcal{H} := (C, f, D, G)$ with modes $\mathcal{Q}$ and transitions $\mathcal{E}$. Let $(t, j) \mapsto \phi(t, j) = \big(q(j), z(t, j)\big)$ be any solution to $\mathcal{H}$ with jump times t_j and $J := \sup_j \mathrm{dom}(\phi)$. For each interval flow $[t_j, t_{j+1}]$ in $\mathrm{dom}(\phi)$, if $j \in \{1, 2, \ldots, J-1\}$, then*

$$\big(q(j), \mathrm{nrv}(z(t_j, j))\big) \xrightarrow{\mathrm{F}} \big(q(j), \mathrm{nrv}(z(t_{j+1}, j))\big) \tag{29}$$

is a flow arrow in $\mathcal{G}$.

Proof. Let $(t, j) \mapsto \phi(t, j) = \big(q(j), z(t, j)\big)$ be a solution to $\mathcal{H}$. Without loss of generality, suppose $J > 1$ (otherwise the conclusion is vacuously true). Take any $j \in \{1, 2, \ldots, J-1\}$. Let $z^{(0)} := z(t_j, j)$, $s^{(0)} := \mathrm{nrv}(z^{(0)})$, $z^{(f)} := z(t_{j+1}, j)$, and $s^{(f)} := \mathrm{nrv}(z^{(f)})$.

Then, $\phi(t_j, j) \in G(\phi(t_j, j-1))$, so $v^{(0)} := (q(j), s^{(0)})$ is a vertex in $\mathcal{V}$. Similarly, $\phi(t_{j+1}, j) \in D$, so $v^{(f)} := (q(j), s^{(f)})$ is in $\mathcal{V}$.

To show that $v^{(0)} \xrightarrow{\mathrm{F}} v^{(f)}$ is a flow arrow, for $\tau := t_{j+1} - t_j$, let $\xi : [0, \tau] \to \mathbb{R}^n$ be defined by

$$t \mapsto \xi(t) := \begin{cases} z(t + t_j, j)/|z^{(0)}| & \text{if } |z^{(0)}| \neq 0 \\ z(t + t_j, j) & \text{if } |z^{(0)}| = 0_n. \end{cases}$$

Since $[t_j, t_{j+1}]$ is an interval of flow, τ is positive. We will check each condition in the flow arrow conditions (21). Equation (21a) is satisfied because $\xi(0) = \mathrm{nrv}(z^{(0)}) = s^{(0)}$. From the flow condition (9b) of hybrid solutions, we have that $\dot{z}(t, j) = f_q(z(t, j))$ for almost all $t \in [t_j, t_{j+1}]$. Since f_q is either constant or linear, $t \mapsto z(j, t)$ is the unique solution to $\dot{x} = f_q(x)$ and $\dot{z}(j, t) = f_q(z(j, t))$ for all $t \in (t_j, t_{j+1})$ (rather than merely *almost all*). Therefore, (21b) is satisfied:

$$\frac{d\xi}{dt}(t) = \frac{dz}{dt}(t + t_j) = f_q(z(t + t_j)) = f_q(\xi(t)) \quad \forall t \in (0, \tau).$$

By (9a), $\phi(t, j) \in C$ for all $t \in (t_j, t_{j+1})$, so $z(t, j) \in C_q$ for all $t \in (t_j, t_{j+1})$ and

$$\xi(t) \in C_q \quad \forall t \in (0, T),$$

satisfying (21c).

Finally, (21d) is satisfied:

$$\begin{aligned} \mathrm{nrv}(\xi(T)) &= \begin{cases} \mathrm{nrv}(z^{(f)}/|z^{(0)}|) & \text{if } |z^{(0)}| \neq 0 \\ \mathrm{nrv}(z^{(f)}) & \text{if } |z^{(0)}| = 0 \end{cases} \\ &= \mathrm{nrv}(z^{(f)}) = s^{(f)}. \end{aligned}$$

Therefore, $v^{(0)} \xrightarrow{\mathrm{F}} v^{(f)}$ is a flow arrow in $\mathcal{G}$. □

Proposition 3. *Consider a conical hybrid system $\mathcal{H}$ with conical transition graph $\mathcal{G}$. For any solution ϕ to $\mathcal{H}$, the $\mathcal{G}$-simulation of ϕ is a well-formed walk through $\mathcal{G}$.*

Proof. Let w be the $\mathcal{G}$-simulation of ϕ, and let $\{K_j\}_{j=0}^{J}$, $\{v_k\}_{k=0}^{K_J}$, and $\{\ell_k\}_{k=0}^{K_J-1}$ be defined as in Definition 8. We write the components of ϕ as $\phi(t, j) = \big(q(j), z(t, j)\big)$. To show that w is a walk through $\mathcal{G}$, we must show that each v_k is a vertex in $\mathcal{V}$, and for each $k \in \{0, 1, \ldots, K_j\}$ that $v_k \to v_{k+1}$ is an arrow in $\mathcal{G}$. The values of K_j always increment by $+1$ or $+2$, i.e., $K_{(j+1)} \in \{K_j+1, K_j+2\}$. Thus, for each $j \in \{0, 1, \ldots, J\}$, we need to show that $v_{K_j} \in \mathcal{V}$ and (if $j < J$) that $v_{K_j+1} \in \mathcal{V}$. Furthermore, we need to show $v_{K_j} \xrightarrow{\ell_{K_j}} v_{K_j+1}$ is an arrow in $\mathcal{A}$. If $K_{(j+1)} = K_j+2$, we also need to show $v_{K_j+1} \xrightarrow{\ell_{K_j+1}} v_{K_j+2}$ is an arrow in $\mathcal{A}$. We will consider separately the cases of $K_{(j+1)} = K_j + 1$ and $K_{(j+1)} = K_j + 2$.

Take any $j \in \{0, 1, \ldots, J\}$.

Case 1 ($K_{(j+1)} = K_j + 1$)**.** Suppose $K_{(j+1)} = K_j + 1$, which requires that either $j = 0$ or $t_j = t_{j+1}$. For the case where $j = 0$, there is a jump arrow in $\mathcal{G}$ from $v_0 = \big(q(0), \operatorname{nrv}(z(t_1, 0))\big)$ to $v_1 = \big(q(1), \operatorname{nrv}(t_1, 1)\big)$ per Lemma 2, since t_1 is a jump time in $\operatorname{dom}(\phi)$. Similarly, if $t_j = t_{j+1}$, then

$$\big(q(j), \operatorname{nrv}(z(t_j, j))\big) = \big(q(j), \operatorname{nrv}(z(t_{j+1}, j))\big) \in D,$$

so, again by Lemma 2, there is a jump arrow in $\mathcal{G}$ from

$$v_{K_j} = \big(q(j), \operatorname{nrv}(z(t_j, j))\big) \quad \text{to} \quad v_{K_{(j+1)}} = \big(q(j+1), \operatorname{nrv}(t_{j+1}, j+1)\big).$$

Case 2 ($K_{(j+1)} = K_j + 2$)**.** Suppose $K_{(j+1)} = K_j + 2$. From the definition of K_j, it follows that $j \in \{1, 2, \ldots, J-1\}$ and $[t_j, t_{j+1}]$ is an interval of flow in $\operatorname{dom}(\phi)$. By Lemma 3, there is a flow arrow in $\mathcal{G}$ from $v_{K_j} = \big(q(j), \operatorname{nrv}(z(t_j, j))\big)$ to $v_{K_j+1} = \big(q(j), z(t_{j+1}, j)\big)$. Additionally, because t_{j+1} is a jump time, there is a jump arrow in $\mathcal{G}$ from

$$v_{(K_j+1)} \quad \text{to} \quad v_{(K_j+2)} = v_{K_{(j+1)}} = \big(q(j+1), \operatorname{nrv}(t_{j+1}, j+1)\big),$$

per Lemma 2.

Therefore, we have shown that each v_k is a vertex in $\mathcal{V}$ and each step in w is an arrow in $\mathcal{A}$, so w is a walk through $\mathcal{G}$. Furthermore, each flow arrow in w is followed by a jump arrow, as shown in Case 2, so w is well-formed. □

Proposition 4. *Consider a conical hybrid system $\mathcal{H}$ with modes $\mathcal{Q}$ and conical transition graph $\mathcal{G}$. For some $K \in \{1, 2, \ldots, \infty\}$, suppose that*

$$w := (v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \cdots \xrightarrow{\ell_{(K-1)}} v_K)$$

is a well-formed walk through $\mathcal{G}$ with $\ell_0 = \text{J}$ and if $K < \infty$, then $\ell_{(K-1)} = \text{J}$. Then, there exists a solution ϕ to $\mathcal{H}$ such that w is the $\mathcal{G}$-simulation of ϕ.

Proof. Let $\tilde{J}$ be the total number of jump arrows in w. For each finite $\tilde{\jmath} \in \{0, 1, \ldots, \tilde{J}\}$, let $K_{\tilde{\jmath}}$ be the index of the vertex in w immediately after $\tilde{\jmath}$-many jump arrows. That is, $K_j \in \mathbb{N}$ is the smallest number such that there are $\tilde{\jmath}$ jump labels in $\{\ell_0, \ell_1, \ldots, \ell_{K_{\tilde{\jmath}}}\}$.

For each finite $k \in \{0, 1, 2, \ldots, K\}$, let $(q_k, s_k) := v_k$. We will construct a sequence $\{\phi_{\tilde{\jmath}}\}_{\tilde{\jmath}=1}^{\tilde{J}}$ of hybrid arcs in the form

$$(t, j) \mapsto \phi_{\tilde{\jmath}}(t, j) = \big(p_{\tilde{\jmath}}(j), z_{\tilde{\jmath}}(t, j)\big), \tag{30}$$

where $\operatorname{dom}(\phi_{\tilde{\jmath}})$ and $z_{\tilde{\jmath}}$ are defined below, and $j \mapsto p_{\tilde{\jmath}}(j) := q_{K_j}$ for each $j \in \{0, 1, \ldots, \tilde{\jmath}\}$. By induction, we will show that for each $\tilde{\jmath} \in \{1, 2, \ldots, \tilde{J}\}$,

(S1) if $\tilde{\jmath} > 1$, then $\phi_{\tilde{\jmath}}$ is an extension of $\phi_{\tilde{\jmath}-1}$,

(S2) $\phi_{\tilde{\jmath}}$ is a solution to $\mathcal{H}$ that jumps $\tilde{\jmath}$ times (i.e., $\sup_j \operatorname{dom}(\phi_{\tilde{\jmath}}) = \tilde{\jmath}$), (S3) $\mathrm{nrv}\big(z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})\big) = s_{K_{\tilde{\jmath}}}$, where $T_{\tilde{\jmath}} := \sup_t \mathrm{dom}(\phi_{\tilde{\jmath}})$ is finite (and $\tilde{\jmath} = \sup_j \mathrm{dom}(\phi_{\tilde{\jmath}})$),

(S4) $w_{\tilde{\jmath}} := (v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \cdots \xrightarrow{\ell_{(K_{\tilde{\jmath}}-1)}} v_{K_{\tilde{\jmath}}})$ is the $\mathcal{G}$-simulation of $\phi_{\tilde{\jmath}}$.

The following definition is used to construct $\phi_{\tilde{\jmath}}$. For each $k \in \{0, 1, \ldots, K-1\}$ such that $\ell_k = \mathrm{F}$, take $\tau_k > 0$ and $\xi_k : [0, \tau_k] \to \mathbb{R}^n$ that satisfy the flow arrow conditions in (21).

For the base case ($\tilde{\jmath} = 1$), let $\mathrm{dom}(\phi_1) := \{(0,0), (0,1)\}$, $z_1(0,0) := s_0$, and $z_1(0,1) := A_{e_0} s_0$. Hence, $\phi_1(0,0) = (q_0, s_0) = v_0$ and $\phi_1(0,1) = \big(q_1, A_{e_0} s_0\big)$. Condition (S1) is vacuously satisfied because $\tilde{\jmath} = 1$. Since $v_0 \xrightarrow{\mathrm{J}} v_1$ is a jump arrow, $\phi_1(0,0)$ is in D. Thus, ϕ_1 is a solution to $\mathcal{H}$ with one jump—satisfying (S2)—because $\mathrm{dom}(\phi_1)$ has no intervals of flow and satisfies (8) at $t_1 = 0$, the only jump time in $\mathrm{dom}(\phi_1)$. Additionally, since $(q_0, s_0) \xrightarrow{\mathrm{J}} (q_1, s_1)$ is a jump arrow, (19) requires that $s_1 = \mathrm{nrv}(A_{e_0} s_0)$. Thus, $\mathrm{nrv}(z_1(T_1, J_1)) = \mathrm{nrv}(z_1(0,1)) = s_1 = s_{K_1}$, satisfying (S3). The walk $w_1 = v_0 \xrightarrow{\mathrm{J}} v_1$ is the $\mathcal{G}$-simulation of ϕ_1 with $h_0 = (0,0)$ and $h_1 = (0,1)$, as defined in Definition 8, thus (S4) is satisfied, finishing the proof that the base case satisfies (S1)–(S4).

For the inductive case, take any $\tilde{\jmath} \in \{1, 2, \ldots, \tilde{J} - 1\}$ and suppose that $\phi_{\tilde{\jmath}}$ is a hybrid arc that satisfies (S1)–(S4). We define $\phi_{\tilde{\jmath}+1}$ as an extension of $\phi_{\tilde{\jmath}}$, i.e., $\mathrm{dom}(\phi_{\tilde{\jmath}}) \subset \mathrm{dom}(\phi_{\tilde{\jmath}+1})$ and $\phi_{\tilde{\jmath}+1}(t, j) := \phi_{\tilde{\jmath}}(t, j)$ for all $(t, j) \in \mathrm{dom}(\phi_{\tilde{\jmath}})$, so (S1) holds by construction. We define $\phi_{\tilde{\jmath}+1}$ beyond the domain of $\phi_{\tilde{\jmath}}$ via three cases. In each case, we will define $k^{\ominus}$ and $k^{\oplus}$ and, for Cases 2 and 3, we also define $k^{(0)}$ and $k^{(\mathrm{f})}$. For the given definitions of $k^{\ominus}$, $k^{\oplus}$, $k^{(0)}$, and $k^{(\mathrm{f})}$, let

$$v^{\circledast} := v_{k^{\circledast}}, \; s^{\circledast} := s_{k^{\circledast}}, \text{ and } q^{\circledast} := q_{k^{\circledast}} \text{ for each } \circledast \in \{\ominus, \oplus, (0), (\mathrm{f})\},$$

and $e := (q_{k^{\ominus}}, q_{k^{\oplus}})$.

Case 1 (jump arrow). Suppose $\ell_{K_{\tilde{\jmath}}}$ is a jump label. There are, $\tilde{\jmath}$-many jump arrows from v_0 to $v_{K_{\tilde{\jmath}}}$ (by the definition of $K_{\tilde{\jmath}}$) and it takes one additional step $v_{K_{\tilde{\jmath}}} \xrightarrow{\mathrm{J}} v_{K_{\tilde{\jmath}}+1}$ for the walk $w_{(\tilde{\jmath}+1)}$ to contain $\tilde{\jmath}+1$ jump arrows, because $\ell_{K_{\tilde{\jmath}}} = \mathrm{J}$, so $K_{(\tilde{\jmath}+1)} = K_{\tilde{\jmath}} + 1$. Let $k^{\ominus} := K_{\tilde{\jmath}}$ and $k^{\oplus} := K_{(\tilde{\jmath}+1)}$, We also define $r_0 := |z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})|$ and $r_{\mathrm{J}} := |A_e s^{(0)}| \in \mathcal{W}(v^{\ominus} \xrightarrow{\mathrm{J}} v^{\oplus})$. Let

$$\mathrm{dom}(\phi_{\tilde{\jmath}+1}) := \mathrm{dom}(\phi_{\tilde{\jmath}}) \cup (\{T_{\tilde{\jmath}}\} \times \{\tilde{\jmath}, \tilde{\jmath}+1\}),$$

and

$$z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}}, \tilde{\jmath}+1) := r_0 A_e s^{\ominus}.$$

Since $v^{\ominus} \xrightarrow{\mathrm{J}} v^{\oplus}$ is a jump arrow in $\mathcal{A}$, we have that $(q^{\ominus}, s^{\ominus}) \in D$. By property (3) of the nrv function and (S3) from the inductive hypothesis,

$$z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath}) = |z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})| \, \mathrm{nrv}(z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})) = r_0 s^{\ominus}.$$

Since D_e is a cone containing $s^{\ominus}$, we have that $z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath}) \in D_e$ and thus $\phi_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})$ is in D. Additionally,

$$\phi_{\tilde{\jmath}+1}(T_{\tilde{\jmath}}, \tilde{\jmath}+1) = \big(q^{\oplus}, A_e(r_0 s^{\ominus})\big) \in G(\phi_{\tilde{\jmath}+1}(T_{\tilde{\jmath}}, \tilde{\jmath})).$$

Therefore, $\phi_{\tilde{\jmath}+1}$ is a solution that jumps $\tilde{\jmath}+1$ times, thereby satisfying (S2). At the end of $\phi_{\tilde{\jmath}+1}$, we have $z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1) = r_0 A_e s^{\ominus}$. By the definition of a jump arrow in (19), $s^{\oplus} = \operatorname{nrv}(A_e s^{\ominus})$. Furthermore, $s^{\oplus} = \operatorname{nrv}(r_0 A_e s^{\ominus}) = \operatorname{nrv}(z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1))$ because $r_0 \geq 0$ with $r_0 = 0$ if and only if $s^{\ominus} = 0_n$, in which case $A_e s^{\ominus} = 0_n$, also. Thus, (S3) is satisfied.

By (S4) in the inductive hypothesis, $w_{\tilde{\jmath}}$ is the CTG-simulation of $\phi_{\tilde{\jmath}}$, so Equations (26) and (27) are satisfied up to $K_{\tilde{\jmath}}$ and $K_{\tilde{\jmath}} - 1$, respectively. For $\tilde{\jmath}+1$, the hybrid times $h_0, h_1, \ldots, h_{K_{\tilde{\jmath}}}$ defined in Definition 8 are the same as for $\tilde{\jmath}$ and $h_{K_{\tilde{\jmath}+1}} = (T_{\tilde{\jmath}+1}, \tilde{\jmath}+1)$. Using (S3), we find that

$$v^{\oplus} = \big(q^{\oplus}, s^{\oplus}\big) = \Big(p_{\tilde{\jmath}+1}(\tilde{\jmath}+1), \operatorname{nrv}\big(z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1)\big)\Big),$$

satisfying (26) for $k = K_{\tilde{\jmath}+1}$. Finally, $\pi_{\text{J}}\big(h_{K_{\tilde{\jmath}+1}}\big) > \pi_{\text{J}}\big(h_{K_{\tilde{\jmath}}}\big)$, so (27) is satisfied for $k = K_{\tilde{\jmath}+1} - 1$. Therefore, $w_{\tilde{\jmath}+1}$ is the CTG-simulation of $\phi_{\tilde{\jmath}+1}$, as required by (S4).

Case 2 (flow arrow in mode with linear flows). Suppose $\ell_{K_{\tilde{\jmath}}} = \text{F}$ and $q_{K_{\tilde{\jmath}}}$ is a mode with linear flows. Since $\ell_{K_{\tilde{\jmath}}} = \text{F}$ and w is well-formed, $\ell_{K_{\tilde{\jmath}}+1}$ is a jump label, so $K_{(\tilde{\jmath}+1)} = K_{\tilde{\jmath}} + 2$. Let $k^{(0)} := K_{\tilde{\jmath}}$, $k^{(\text{f})} := K_{\tilde{\jmath}} + 1$, $k^{\ominus} := K_{\tilde{\jmath}} + 1$, and $k^{\oplus} := K_{\tilde{\jmath}} + 2$. From the definition of flow arrows, take $\tau > 0$ and $\xi : [0, \tau] \to \mathbb{R}^n$ that satisfy (21) for $v^{(0)} \xrightarrow{\text{F}} v^{(\text{f})}$. We also define $q := q^{(0)} = q^{(\text{f})}$, $r_0 := |z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})|$, $r_{\text{F}} := |\xi(\tau)| \in \mathcal{W}(v^{(0)} \xrightarrow{\text{F}} v^{(\text{f})})$, and $r_{\text{J}} := |A_e s^{\ominus}| \in \mathcal{W}(v^{(0)} \xrightarrow{\text{J}} v^{(\text{f})})$. The extension of $\operatorname{dom}(\phi_{\tilde{\jmath}})$ is defined as

$$\operatorname{dom}(\phi_{\tilde{\jmath}+1}) := \operatorname{dom}(\phi_{\tilde{\jmath}}) \cup ([T_{\tilde{\jmath}}, T_{\tilde{\jmath}} + \tau] \times \{\tilde{\jmath}, \tilde{\jmath}+1\}).$$

Thus, $T_{\tilde{\jmath}+1} = T_{\tilde{\jmath}} + \tau$. The values of $z_{\tilde{\jmath}+1}$ are defined for $(t, j) \in \operatorname{dom}(\phi_{\tilde{\jmath}+1}) \setminus \operatorname{dom}(\phi_{\tilde{\jmath}})$ as

$$z_{\tilde{\jmath}+1}(t, \tilde{\jmath}) := r_0 \xi(t - T_{\tilde{\jmath}}) \quad \forall t \in [T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1}] \tag{31a}$$

$$z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1) := r_0 r_{\text{F}} A_e s^{\ominus}. \tag{31b}$$

Since $\xi(t) \in C_q$ for all $t \in (0, \tau)$ and C_q is a cone, $z_{\tilde{\jmath}+1}(t, \tilde{\jmath})$ is also in C_q for all $t \in (T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1})$, satisfying (9a). Furthermore, the hybrid arc $\phi_{\tilde{\jmath}+1}$ satisfies the flow condition (9b) for all $t \in (T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1})$:

$$\begin{aligned} \frac{dz_{\tilde{\jmath}+1}}{dt}(t, j) &= \frac{d}{dt}\big(r_0 \xi\big(t - T_{\tilde{\jmath}}\big)\big) = r_0 f_q(\xi(t - T_{\tilde{\jmath}})) = r_0 A_q \xi(t - T_{\tilde{\jmath}}) \\ &= A_q r_0 \xi(t - T_{\tilde{\jmath}}) = f_q(z_{\tilde{\jmath}+1}(t, j)). \end{aligned}$$

By the definition of flow arrows, namely (21d), $\operatorname{nrv}(\xi(\tau)) = s^{(\text{f})}$, so at the end of the interval of flow $(T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1})$, we have $z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) = r_0 \xi(\tau) \in D_e$. Thus, $\phi_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) \in D$, satisfying (8a). Furthermore, (8b) is satisfied because $\phi_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) \in G(\phi_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}))$ because

$$\begin{aligned} z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1) &= r_0 r_{\text{F}} A_e s^{\ominus} = A_e(r_0 r_f s^{\ominus}) = A_e(r_0 |\xi(\tau)| \operatorname{nrv}(\xi(\tau))) \\ &= A_e(r_0 \xi(\tau)) = A_e z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}). \end{aligned} \tag{32}$$

Therefore, $\phi_{\tilde{\jmath}+1}$ is a solution to $\mathcal{H}$ that jumps one more time than $\phi_{\tilde{\jmath}}$, satisfying (S2).

Let $z_0 := z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}}, J_{\tilde{\jmath}})$ and $z_f := z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, J_{\tilde{\jmath}+1})$. We want to show $\mathrm{nrv}(z_f) = s^{\oplus} = s_{K_{\tilde{\jmath}}+1}$. By definition, in (32), $z_f = r_0 r_{\mathrm{F}} A_e s^{\ominus}$. If $r_0 r_{\mathrm{F}} > 0$, then we have that $\mathrm{nrv}(z_f) = s^{\oplus}$, per (4), since $s^{\oplus} = \mathrm{nrv}(A_e s^{\ominus})$.

On the other hand, if $r_0 = 0$, then $s^{(\mathrm{f})} = s^{\ominus} = z_0 = 0_n$ and, because mode q has linear flows, solutions cannot flow away from the origin, so $r_{\mathrm{F}} = 0$, $z_f = 0_n$, and $s^{(\mathrm{f})} = s^{\ominus} = 0_n$, so $s^{\oplus} = \mathrm{nrv}(A_e s^{\ominus}) = \mathrm{nrv}(0_n) = 0_n$. Thus, $\mathrm{nrv}(z_f) = s^{\oplus}$.

Finally, if $r_{\mathrm{F}} = 0$, then $s^{(\mathrm{f})} = s^{\ominus} = 0_n$, because $|\xi(\tau)| = 0$. Since $s^{\oplus} = \mathrm{nrv}(A_e s^{\ominus}) = 0_n$, we have that

$$\mathrm{nrv}(z_f) = s^{\oplus} = 0_n.$$

Next, we want to show (S4), i.e., that $w_{\tilde{\jmath}+1}$ is the $\mathcal{G}$-simulation of $\phi_{\tilde{\jmath}+1}$, which requires showing (26) holds for $k \in \{K_{\tilde{\jmath}}+1, K_{\tilde{\jmath}}+2\}$, and (27) holds for $k \in \{K_{\tilde{\jmath}}, K_{\tilde{\jmath}}+1\}$. By assumption, $w_{\tilde{\jmath}}$ is the $\mathcal{G}$-simulation of $z_{\tilde{\jmath}}$. For $z_{\tilde{\jmath}+1}$, the sequence $h_0, h_1, \ldots, h_{K_{(\tilde{\jmath}+1)}}$ defined in Definition 8 has two more elements than the corresponding sequence for $z_{\tilde{\jmath}}$, namely $h_{(K_{\tilde{\jmath}}+1)} = (T_{\tilde{\jmath}+1}, \tilde{\jmath})$ and $h_{(K_{\tilde{\jmath}}+2)} = h_{K_{(\tilde{\jmath}+1)}} = (T_{\tilde{\jmath}+1}, \tilde{\jmath}+1)$.

First, we will show that $w_{\tilde{\jmath}+1}$ satisfies (26) for $k = K_{\tilde{\jmath}}+1$. We have that

$$p_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = p_{\tilde{\jmath}+1}(\tilde{\jmath}) = q_{K_{\tilde{\jmath}}}$$

because $p_{\tilde{\jmath}+1}(j) = q_{K_j}$ by definition for each $j \in \{0, 1, \ldots, \tilde{\jmath}+1\}$. But,

$$p_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = q_{K_{\tilde{\jmath}}+1},$$

because $q = q^{(0)} = q_{K_{\tilde{\jmath}}} = q_{K_{\tilde{\jmath}}+1} = q^{(\mathrm{f})}$, as required by (26) for $k = K_{\tilde{\jmath}}+1$. For the z-component,

$$z_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) = r_0 \xi(\tau).$$

Suppose, first, that $r_0 > 0$. Then, by (4), Suppose, instead, that $r_0 = 0$. In this case, ξ is identically zero because $t \mapsto \xi(t) := 0_n$ is the unique solution to $\dot{x} = A_q x$ from $x_0 = 0_n$. Thus, $\mathrm{nrv}(r_0 \xi(\tau)) = \mathrm{nrv}(\xi(\tau)) = 0_n$. By (21d), $\mathrm{nrv}(\xi(\tau)) = s^{(\mathrm{f})}$, so

$$z_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = s_{K_{\tilde{\jmath}}+1},$$

therefore (26) is satisfied for $k = K_{\tilde{\jmath}}+1$.

Next, we will show that $w_{\tilde{\jmath}+1}$ satisfies (26) for $k = K_{\tilde{\jmath}}+2 = K_{(\tilde{\jmath}+1)}$. We have that

$$p_{\tilde{\jmath}+1}(h_{K_{(\tilde{\jmath}+1)}}) = p_{\tilde{\jmath}+1}(\tilde{\jmath}+1) = q_{K_{(\tilde{\jmath}+1)}},$$

as required by (26) for $k = K_{\tilde{\jmath}}+1$. For the z-component,

$$z_{\tilde{\jmath}+1}(h_{K_{(\tilde{\jmath}+1)}}) = z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1).$$

We have already shown that (S3) holds, so $\mathrm{nrv}(z_{\tilde{\jmath}+1}(h_{K_{(\tilde{\jmath}+1)}})) = s_{K_{(\tilde{\jmath}+1)}}$. Therefore, (26) holds for $k = K_{(\tilde{\jmath}+1)}$.

Finally, (27) is satisfied for $k = K_{\tilde{\jmath}}$ because $\ell_{K_{\tilde{\jmath}}} = \mathrm{F}$ and

$$\pi_{\mathrm{T}}(h_{K_{\tilde{\jmath}}+1}) > \pi_{\mathrm{T}}(h_{K_{\tilde{\jmath}}}),$$

and is satisfied for $k = K_{\tilde{\jmath}} + 1$ because $\ell_{K_{\tilde{\jmath}}+1} = \text{J}$ and

$$\pi_{\text{J}}\big(h_{K_{\tilde{\jmath}+1}}\big) > \pi_{\text{J}}\big(h_{K_{\tilde{\jmath}}+1}\big).$$

Therefore, $w_{\tilde{\jmath}+1}$ is the $\mathcal{G}$-simulation of $\phi_{\tilde{\jmath}+1}$, satisfying (S4).

Case 3 (flow arrow in mode with constant flows). Suppose $\ell_{K_{\tilde{\jmath}}} = \text{F}$ and $q_{K_{\tilde{\jmath}}}$ is a mode with constant flows. Since w is well-formed, $\ell_{K_{\tilde{\jmath}}+1} = \text{J}$ and $K_{(\tilde{\jmath}+1)} = K_{\tilde{\jmath}} + 2$. Let $k^{(0)} := K_{\tilde{\jmath}}$, $k^{(\text{f})} := K_{\tilde{\jmath}} + 1$, $k^{\ominus} := K_{\tilde{\jmath}} + 1$, and $k^{\oplus} := K_{\tilde{\jmath}} + 2$. From the definition of flow arrows, take $\tau > 0$ and $\xi : [0, \tau] \to \mathbb{R}^n$ that satisfy (21) for $v^{(0)} \xrightarrow{\text{F}} v^{(\text{f})}$. We also define $q := q^{(0)} = q^{(\text{f})}$, $r_0 := |z_{\tilde{\jmath}}(T_{\tilde{\jmath}}, \tilde{\jmath})|$, $r_{\text{F}} := |\xi(\tau)| \in \mathcal{W}(v^{(0)} \xrightarrow{\text{F}} v^{(\text{f})})$, and $r_{\text{J}} := |A_e s^{\ominus}| \in \mathcal{W}(v^{(0)} \xrightarrow{\text{J}} v^{(\text{f})})$. Let

$$\operatorname{dom}(\phi_{\tilde{\jmath}+1}) := \operatorname{dom}(\phi_{\tilde{\jmath}}) \cup \big([T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1}] \times \{\tilde{\jmath}, \tilde{\jmath}+1\}\big),$$

where

$$T_{\tilde{\jmath}+1} := T_{\tilde{\jmath}} + \begin{cases} r_0 \tau & \text{if } r_0 > 0 \\ \tau & \text{if } r_0 = 0. \end{cases}$$

The value of $z_{\tilde{\jmath}+1}(t, j)$ is defined for $(t, j) \in \operatorname{dom}(\phi_{\tilde{\jmath}+1}) \setminus \operatorname{dom}(\phi_{\tilde{\jmath}})$ as

$$z_{\tilde{\jmath}+1}(t, \tilde{\jmath}) := \begin{cases} r_0 \xi\big((t - T_{\tilde{\jmath}})/r_0\big) & \text{if } r_0 > 0 \\ \xi\big(t - T_{\tilde{\jmath}}\big) & \text{if } r_0 = 0 \end{cases} \quad \forall t \in [T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1}] \tag{33a}$$

$$z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1) := \begin{cases} r_0 r_{\text{F}} A_e s^{\ominus} & \text{if } r_0 > 0 \\ r_{\text{F}} A_e s^{\ominus} & \text{if } r_0 = 0. \end{cases} \tag{33b}$$

To show (S2), we will consider separately the cases $r_0 > 0$ and $r_0 = 0$. Suppose, first, that $r_0 > 0$. Then, $\phi_{\tilde{\jmath}+1}$ satisfies the flow condition (9b) for all $t \in (T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1})$:

$$\begin{aligned} \dot{z}_{\tilde{\jmath}+1}(t, \tilde{\jmath}) &= \frac{d}{dt}\big(r_0 \xi\big((t - T_{\tilde{\jmath}})/r_0\big)\big) \\ &= r_0 f_q\big(\xi\big((t - T_{\tilde{\jmath}})/r_0\big)\big) \frac{d}{dt}\big((t - T_{\tilde{\jmath}})/r_0\big) \\ &= f_q(z_{\tilde{\jmath}+1}(t, \tilde{\jmath})). \end{aligned}$$

If, instead, $r_0 = 0$, then,

$$\dot{z}_{\tilde{\jmath}+1}(t, \tilde{\jmath}) = \frac{d}{dt}\big(\xi\big(t - T_{\tilde{\jmath}}\big)\big) = f_q\big(z_{\tilde{\jmath}+1}(t, \tilde{\jmath})\big),$$

again satisfying (9b). In both cases, $z_{\tilde{\jmath}+1}(t, \tilde{\jmath}) \in C_q$ for all $t \in (T_{\tilde{\jmath}}, T_{\tilde{\jmath}+1})$, satisfying (9a).

By (21d) in the definition of flow arrows, $\operatorname{nrv}(\xi(\tau)) = s^{(\text{f})} \in D_e$, so

$$z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) \in D_e.$$

Thus, $\phi_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) \in D$, satisfying (8a). Furthermore, (8b) is satisfied at the only jump time, $T_{\tilde{\jmath}+1}$, in $\operatorname{dom}(\phi_{\tilde{\jmath}+1}) \setminus \operatorname{dom}(\phi_{\tilde{\jmath}})$ because $z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1) = A_e z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath})$. Therefore, $\phi_{\tilde{\jmath}+1}$ is a solution to $\mathcal{H}$ that jumps one more time than $\phi_{\tilde{\jmath}}$, satisfying (S2).

Let $z_0 := z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}}, J_{\tilde{\jmath}})$ and $z_f := z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, J_{\tilde{\jmath}+1})$. To prove (S3) holds, we must show $\mathrm{nrv}(z_f) = s^{\oplus} = s_{K_{\tilde{\jmath}}+1}$. From (33b), $\mathrm{nrv}(z_f) = \mathrm{nrv}(A_e s^{\ominus})$, which equals $s^{\oplus}$, satisfying (S3).

Next, we want to show (S4), i.e., that $w_{\tilde{\jmath}+1}$ is the $\mathcal{G}$-simulation of $\phi_{\tilde{\jmath}+1}$, which requires showing (26) holds for $k \in \{K_{\tilde{\jmath}}+1, K_{\tilde{\jmath}}+2\}$, and (27) holds for $k \in \{K_{\tilde{\jmath}}, K_{\tilde{\jmath}}+1\}$. By assumption, $w_{\tilde{\jmath}}$ is the $\mathcal{G}$-simulation of $z_{\tilde{\jmath}}$. For $z_{\tilde{\jmath}+1}$, the sequence $h_0, h_1, \ldots, h_{K_{(\tilde{\jmath}+1)}}$ defined in Definition 8 has two more elements than the corresponding sequence for $z_{\tilde{\jmath}}$, namely $h_{(K_{\tilde{\jmath}}+1)} = (T_{\tilde{\jmath}+1}, \tilde{\jmath})$ and $h_{(K_{\tilde{\jmath}}+2)} = h_{K_{(\tilde{\jmath}+1)}} = (T_{\tilde{\jmath}+1}, \tilde{\jmath}+1)$. First, we will show that $w_{\tilde{\jmath}+1}$ satisfies (26) for $k = K_{\tilde{\jmath}}+1$. We have that

$$p_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = p_{\tilde{\jmath}+1}(\tilde{\jmath}) = q_{K_{\tilde{\jmath}}}$$

because $p_{\tilde{\jmath}+1}(j) = q_{K_j}$ by definition for each $j \in \{0, 1, \ldots, \tilde{\jmath}+1\}$. But,

$$p_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = q_{K_{\tilde{\jmath}}+1},$$

because $q = q^{(0)} = q_{K_{\tilde{\jmath}}} = q_{K_{\tilde{\jmath}}+1} = q^{(\mathrm{f})}$, thereby satisfying (26) for $k = K_{\tilde{\jmath}}+1$. For the z-component,

$$z_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}) = \begin{cases} r_0 \xi(\tau) & \text{if } r_0 > 0 \\ \xi(\tau) & \text{if } r_0 = 0 \end{cases}$$

Thus, $\mathrm{nrv}(z_{\tilde{\jmath}+1}) = \mathrm{nrv}(\xi(\tau))$. By (21d), $\mathrm{nrv}(\xi(\tau)) = s^{(\mathrm{f})}$, so

$$z_{\tilde{\jmath}+1}(h_{K_{\tilde{\jmath}}+1}) = s_{K_{\tilde{\jmath}}+1},$$

therefore (26) is satisfied for $k = K_{\tilde{\jmath}}+1$.

Next, we will show that $w_{\tilde{\jmath}+1}$ satisfies (26) for $k = K_{\tilde{\jmath}}+2 = K_{(\tilde{\jmath}+1)}$. We have that

$$p_{\tilde{\jmath}+1}(h_{K_{(\tilde{\jmath}+1)}}) = p_{\tilde{\jmath}+1}(\tilde{\jmath}+1) = q_{K_{(\tilde{\jmath}+1)}},$$

as required by (26) for $k = K_{\tilde{\jmath}}+1$. For the z-component,

$$z_{\tilde{\jmath}+1}(h_{K_{(\tilde{\jmath}+1)}}) = z_{\tilde{\jmath}+1}(T_{\tilde{\jmath}+1}, \tilde{\jmath}+1).$$

We have already shown that (S3) holds, so $\mathrm{nrv}(z_{\tilde{\jmath}+1}(h_{K_{(\tilde{\jmath}+1)}})) = s_{K_{(\tilde{\jmath}+1)}}$. Therefore, (26) holds for $k = K_{(\tilde{\jmath}+1)}$.

Finally, (27) is satisfied for $k = K_{\tilde{\jmath}}$ because $\ell_{K_{\tilde{\jmath}}} = \mathrm{F}$ and

$$\pi_{\mathrm{T}}\big(h_{K_{\tilde{\jmath}}+1}\big) > \pi_{\mathrm{T}}\big(h_{K_{\tilde{\jmath}}}\big),$$

and is satisfied for $k = K_{\tilde{\jmath}}+1$ because $\ell_{K_{\tilde{\jmath}}+1} = \mathrm{J}$ and

$$\pi_{\mathrm{J}}\big(h_{K_{\tilde{\jmath}+1}}\big) > \pi_{\mathrm{J}}\big(h_{K_{\tilde{\jmath}}+1}\big).$$

Therefore, $w_{\tilde{\jmath}+1}$ is the $\mathcal{G}$-simulation of $\phi_{\tilde{\jmath}+1}$, satisfying (S4).

Thus, for the inductive case, $\phi_{\tilde{\jmath}+1}$ is a hybrid arc that satisfies (S1)–(S4). Therefore, by induction, $w_J = w$ is the $\mathcal{G}$-simulation of $\phi_J = \phi$. □

The following result asserts that the weight of a solution ϕ's CTG-simulation contains the relative change in the distance of ϕ from $\mathcal{Q} \times \{0_n\}$. In other words, the weights of CTG-simulations tell us how solutions move toward or away from $\mathcal{Q} \times \{0_n\}$.

Proposition 5. *Consider a conical hybrid system $\mathcal{H}$ with conical transition graph $\mathcal{G}$ and a solution*

$$(t, j) \mapsto \phi(t, j) = \big(q(j), z(t, j)\big).$$

Suppose ϕ jumps at least once and let

$$w := (v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \cdots \xrightarrow{\ell_{(K_J-1)}} v_{K_J})$$

be the $\mathcal{G}$-simulation of ϕ, with $J := \sup_j \operatorname{dom}(\phi)$ and with $K_0, K_1, \ldots, K_J$ defined as in Definition 8. Furthermore, let h_0, h_1, ..., h_{K_J} be the hybrid times associated with the $\mathcal{G}$-simulation of ϕ, as defined in Definition 8, and for each finite $k \in \{0, 1, \ldots, K_J\}$, let

$$w_k := (v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \cdots \xrightarrow{\ell_{(k-1)}} v_k)$$

be the truncation of w to the first k arrows.

Suppose that there does not exist a flow arrow $(q, 0_n) \xrightarrow{\text{F}} (q, s^{(f)})$ in w such that $s^{(f)} \neq 0_n$. Then, for each finite $k \in \{1, 2, \ldots, K_J\}$,

$$|z(h_k)| = r_k |z(h_0)| \text{ for some } r_k \in \mathcal{W}(w_k). \tag{34}$$

Proof. For each $k \in \{0, 1, \ldots, K_J\}$, let $(q_k, s_k) := v_k$, and if $\ell_k = \text{J}$, then let $e_k := (q_k, q_{k+1})$. Let $r_0 := |z(h_0)|$.

We proceed by induction over k. For the base case, consider $k = 1$. The first arrow in a CTG-simulation is always a jump arrow. Let $r_1 := |A_{e_0} s_0| \in \mathcal{W}(v_0 \xrightarrow{\text{J}} v_1)$. Since $z(t_1, 1) = A_{e_0} z(h_0)$ and $z(h_0) = r_0 s_0$, we have

$$|z(h_1)| = |A_{e_0} r_0 s_0| = r_0 r_1 = r_1 |z(h_0)|.$$

Therefore, (34) holds for $k = 1$, proving the base case.

Now, suppose that (34) holds for $k \in \{1, \ldots, K_J - 1\}$. That is, there exists $r_k \in \mathcal{W}(w_k)$ such that $|z(h_k)| = r_k |z(h_0)|$.

Suppose, first, that $\ell_k = \text{J}$. Let $r'_{k+1} := |A_{e_k} s_k| \in \mathcal{W}(v_k \xrightarrow{\text{J}} v_{k+1})$. Thus, $r_{k+1} := r_k r'_{k+1} \in \mathcal{W}(w_{k+1})$. Furthermore, $z(h_{k+1}) = A_{e_k} z(h_k)$, so

$$|z(h_{k+1})| = |A_{e_k} z(h_k)| = |A_{e_k} r_0 r_k s_k| = r_0 r_k |A_{e_k} s_k| = r_0 r_k r'_{k+1} = r_0 r_{k+1}.$$

Thus, $|z(h_{k+1})| = r_{k+1} |z(h_0)|$ for $r_{k+1} \in \mathcal{W}(w_{k+1})$.

Alternatively, suppose that $\ell_k = \text{F}$. Let $q := q_k = q_{k+1}$. If $r_k = 0$, then $s_k = 0_n$, so $s_{k+1} = 0_n$ also, by assumption, in which case $f_q(0_n) = 0_n$. Since f_q is Lipschitz continuous, solutions to $\dot{z} = f_q(z)$, $z(0) = 0_n$ are unique, namely $t \mapsto \xi(t) := 0_n$. Thus, it must be the case that $z(h_{k+1}) = 0_n$. From (22), we have that $0 \in \mathcal{W}(v_k \xrightarrow{\text{F}} v_{k+1})$, so $r_{k+1} := 0 \in \mathcal{W}(w_{k+1})$. Thus, (34) holds:

$$|z(h_{k+1})| = 0 = r_{k+1} r_0 = r_{k+1} |z(h_0)|.$$

Suppose, instead, that $r_k > 0$, which also implies that $r_0 > 0$ (if $r_0 = 0$, then $r_1 = r_2 = \cdots = r_k = 0$). We will define $\tau > 0$ and $\xi : [0, \tau] \to \mathbb{R}^n$ to satisfy the flow arrow conditions in (21) for $v_k \xrightarrow{\text{F}} v_{k+1}$. Let $j := \pi_{\text{J}}(h_k) = \pi_{\text{J}}(h_{k+1})$, $t_k := \pi_{\text{T}}(h_k)$, and $t_{k+1} = \pi_{\text{T}}(h_{k+1})$. We define

$$\tau := \begin{cases} t_{k+1} - t_k & \text{if } q \text{ has linear flows} \\ \big(t_{k+1} - t_k\big)/r_0 r_k & \text{if } q \text{ has constant flows,} \end{cases}$$

and for all $t \in [0, \tau]$, let

$$t \mapsto \xi(t) := \begin{cases} z\big(t_k + t,\ j\big)/r_0 r_k & \text{if } q \text{ has linear flows} \\ z\big(t_k + r_0 r_k t,\ j\big)/r_0 r_k & \text{if } q \text{ has constant flows.} \end{cases}$$

By the inductive hypothesis, $r_0 r_k = |z(h_k)|$, so we find that (21a) is satisfied:

$$\xi(0) = \frac{1}{r_0 r_k} z(t_k, j) = \frac{z(h_k)}{|z(h_k)|} = \operatorname{nrv}(z(h_k)) = s_k.$$

To check that $\dot{\xi}(t) = f_q(\xi(t))$, that is, (21b), we consider constant flows and linear flows separately. If q has linear flows, then for all $t \in (0, \tau)$,

$$\begin{aligned} \dot{\xi}(t) &= \frac{d}{dt}\big(z(t_k + t, j)/r_0 r_k\big) = \frac{1}{r_0 r_k} \dot{z}(t_k + t, j) \\ &= \frac{1}{r_0 r_k} f_q(z(t_k + t, j)) = \frac{1}{r_0 r_k} A_q z(t_k + t, j) \\ &= A_q z(t_k + t, j)/r_0 r_k = A_q \xi(t) \\ &= f_q(\xi(t)). \end{aligned}$$

Alternatively, if q has constant flows, then

$$\begin{aligned} \dot{\xi}(t) &= \frac{d}{dt}\left(\frac{1}{r_0 r_k} z(t_k + r_0 r_k t)\right) \\ &= \dot{z}(t_k + r_0 r_k t) \\ &= f_q\big(z(t_k + r_0 r_k t)\big). \end{aligned}$$

Since f_q is constant, i.e., $f_q(z) = f_q^*$ for all $z \in \mathbb{R}^n$,

$$f_q\big(z(t_k + r_0 r_k t)\big) = f_q^* = f_q(\xi(t)),$$

so $\dot{\xi}(t) = f_q(\xi(t))$. In both cases, (21b) is satisfied.

We have that $\xi(t) \in C_q$ for all $t \in [0, \tau]$, C_q is a cone, so $z(t, j) \in C_q$ for all $t \in [t_k, t_{k+1}]$, satisfying (21c).

Checking the terminal flow arrow condition (21d), we find

$$\begin{aligned}
\xi(\tau) &= \begin{cases} \xi\big(t_{k+1} - t_k\big) & \text{if } q \text{ has linear flows} \\ \xi\big((t_{k+1} - t_k)/r_0 r_k\big) & \text{if } q \text{ has constant flows} \end{cases} \\
&= \begin{cases} z\big(t_k + (t_{k+1} - t_k), j\big)/r_0 r_k & \text{if } q \text{ has linear flows} \\ z\big(t_k + r_0 r_k\, (t_{k+1} - t_k)/r_0 r_k\,, j\big)/r_0 r_k & \text{if } q \text{ has constant flows} \end{cases} \\
&= z\big(t_{k+1}, j\big)/r_0 r_k \\
&= z\big(h_{k+1}\big)/r_0 r_k\,. \qquad (35)
\end{aligned}$$

Therefore, (21d) holds:

$$\operatorname{nrv}(\xi(\tau)) = \operatorname{nrv}\big(\, z(h_{k+1})/r_k r_0 \,\big) = \operatorname{nrv}(z(h_{k+1})) = s_{k+1}.$$

Finally, let $r'_{k+1} := |\xi(\tau)| \in \mathcal{W}(v_k \xrightarrow{\text{F}} v_{k+1})$ and $r_{k+1} := r_k r'_{k+1} \mathcal{W}(w_{k+1})$. Rewriting (35), we find

$$|z(h_{k+1})| = r_0 r_k |\xi(\tau)| = r_0 r_k r'_{k+1} = r_0 r_{k+1} = r_{k+1} |z(h_0)|.$$

Therefore, (34) holds for all $k \in \{1, 2, \dots, K_J\}$, by induction. □

5.2. Stability and Asymptotic Stability

By applying Propositions 3–5, we can use the CTG of $\mathcal{H}$ to determine pre-asymptotic stability of $\mathcal{O}$. First, in Proposition 6, we use the CTG to establish stability, which we use to establish pre-asymptotic stability in Theorem 2.

Proposition 6. *Let $\mathcal{H} = (C, f, D, G)$ by a conical hybrid system with modes $\mathcal{Q}$ and conical transition graph $\mathcal{G}$. Suppose that $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ is stable for (C, f) and that there exists $M \geq 1$ such that $\sup \mathcal{W}(w) \leq M$ for each walk w through $\mathcal{G}$. Then, $\mathcal{O}$ is stable for $\mathcal{H}$.*

Proof. Take any $\varepsilon > 0$. Since $\mathcal{O}$ is stable for (C, f), there exists $\delta \in (0, \varepsilon)$ such that for every solution $t \mapsto \xi(t)$ to (C, f) with $|\xi(0)| \leq \delta$,

$$|\xi(t)| \leq \varepsilon \quad \forall t \in \operatorname{dom}(\xi).$$

Let $\varepsilon' := \delta/M$. Then, again by the stability of 0_n, there exists $\delta' > 0$ such that, for every solution $t \mapsto \xi(t)$ to (C, f) with $|\xi(0)| \leq \delta'$,

$$|\xi(t)| \leq \varepsilon' \quad \forall t \in \operatorname{dom}(\xi).$$

Let $(t, j) \mapsto \phi(t, j) := (q(j), z(t, j))$ be any solution to $\mathcal{H}$ with $|z(0, 0)| \leq \delta'$. Thus, $|z(t, 0)| \leq \varepsilon'$ for all $t \in [0, t_1]$, where t_1 is the first jump time in $\operatorname{dom}(\phi)$. In particular, we will use the fact that

$$|z(t_1, 0)| \leq \varepsilon'.$$

Since $\mathcal{O}$ is stable for (C, f), solutions to $\mathcal{H}$ cannot leave $\mathcal{O}$ by flowing. Furthermore, from the definition of conical hybrid systems, $G(\mathcal{O}) \subset \mathcal{O}$, so solutions to $\mathcal{H}$ cannot jump away from $\mathcal{O}$. Therefore, $\mathcal{O}$ is forward invariant for $\mathcal{H}$.

Let w be the $\mathcal{G}$-simulation of ϕ with K_1, K_2, …, K_J and h_0, h_1, …, h_{K_J} defined as in Definition 8, and let w_k be the truncation of w to the first $k > 0$ steps. By Proposition 5, for each jump time t_j in $\operatorname{dom}(\phi)$, there exists $r_j \in \mathcal{W}(w_{K_j})$ such that

$$|z(t_j, j)| = r_{K_j} |z(t_1, 0)|.$$

Since the weight of every walk is bounded by M and $|z(t_1, 0)| \leq \varepsilon'$,

$$|z(t_j, j)| = r_{K_j} |z(t_1, 0)| \leq M\varepsilon' = \delta.$$

Thus, every interval of flow $[t_j, t_{j+1}]$ starts with $|z(t_j, j)| \leq \delta$, so $|z(t, j)| \leq \varepsilon$ for all $t \in [t_j, t_{j+1}]$. Therefore, $\mathcal{O}$ is stable for $\mathcal{H}$. □

The next theorem is of central importance to this work as it allows one to establish pre-asymptotic stability using the CTG.

Theorem 2. *Let $\mathcal{H} = (C, f, D, G)$ be a conical hybrid system with modes $\mathcal{Q}$ and conical transition graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. Suppose the following:*

(P1) *For each $q \in \mathcal{Q}$, the origin 0_n is pre-asymptotically stable for (C_q, f_q).*

(P2) *There exists $M > 0$ such that every walk w through $\mathcal{G}$ satisfies $\sup \mathcal{W}(w) \leq M$.*

(P3) *Every well-formed infinite-length walk w through $\mathcal{G}$ satisfies $\mathcal{W}(w) = \{0\}$.*

Then, the set $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ is pAS with respect to $\mathcal{H}$.

Proof. Items (P1) and (P2) satisfy the assumptions of Proposition 6, which asserts that the origin of $\mathcal{H}$ is stable. By stability and the radial homogeneity property of $\mathcal{H}$ established in Proposition 1, every solution is bounded. Thus, we only need to show that every complete solution to $\mathcal{H}$ converges to $\mathcal{O}$. As a consequence of stability, $\mathcal{O}$ is forward invariant and there does not exist any flow arrows in $\mathcal{G}$ in the form $(q, 0_n) \xrightarrow{\text{F}} (q, s^{(\text{f})})$, where $s^{(\text{f})} \neq 0_n$.

Let $(t, j) \mapsto \phi(t, j) = (q(j), z(t, j))$ be any solution to $\mathcal{H}$, let $J := \sup_j \operatorname{dom}(\phi)$ and $T := \sup_t \operatorname{dom}(\phi)$, let $t_0 := 0$, and for each $j \in \{1, 2, \ldots, J\}$, let t_j denote the jth jump time of ϕ. Showing that ϕ converges to $\mathcal{O}$ is equivalent to showing z converges to 0_n.

If $J < \infty$, then there are no jumps after t_J, so the function $t \mapsto \phi(t, J)$ is a solution to (C, f) for all $t \in [t_J, T)$. If ϕ is complete, then $\lim_{t \to \infty} z(t, J) = 0_n$ due to (P1).

Suppose, instead, that $J = \infty$. Let w be the $\mathcal{G}$-simulation of ϕ with K_1, K_2, …, K_J; $h_0, h_1, \ldots, h_{K_J}$; and w_k defined as in Definition 8. Then, by Proposition 5, for each finite $k \in \{1, 2, \ldots, K_J\}$, there exists $r_k \in \mathcal{W}(w_k)$ such that

$$|z(h_k)| = r_k |z(t_1, 0)| \leq \sup \mathcal{W}(w_k).$$

Per Proposition 3, the CTG-simulation of ϕ is a well-formed walk through $\mathcal{G}$, so by (P3), we have that $\mathcal{W}(w) = \{0\}$. Thus,

$$\lim_{k\to\infty} \sup \mathcal{W}(w_k) = 0 \quad \text{and} \quad \lim_{k\to\infty} |z(t_j, j)| = 0.$$

It remains to be shown that the value of the solution during intervals of flow *between* jump times also converges. By assumption, $\mathcal{O}$ is stable for (C, f), so the assumptions of Lemma 1 are satisfied. Thus, by Lemma 1, we have that

$$\lim_{t+j\to\infty} z(t, j) = 0_n.$$

Therefore, since every complete solution to $\mathcal{H}$ converges to $\mathcal{O}$, we conclude that $\mathcal{O}$ is pAS for $\mathcal{H}$. □

Remark 4. If, additionally, $\check{\mathcal{H}}$ is the conical approximation of a hybrid system $\mathcal{H}$ about $x_* \in \mathbb{R}^n$, then Theorem 1 asserts that x_* is (locally) pre-asymptotically stable for $\mathcal{H}$.

The assumptions in Theorem 2 can be simplified when $\mathcal{G}$ is finite. When $\mathcal{V}$ is finite, condition (P3) is satisfied if and only if $\sup \mathcal{W}(w) < 1$ for every elementary cycle w in $\mathcal{G}$. A walk through a graph is called an *elementary cycle* if it starts and ends at the same vertex and does not visit any other vertex more than once. To check (P3), it is necessary to enumerate over all of the elementary cycles. One efficient algorithm for this purpose is Johnson's enumeration algorithm [29]. For a CTG with $|\mathcal{V}|$ vertices, $|\mathcal{A}|$ arrows, and c elementary circuits (not counting cyclic permutations), the worst-case time complexity of Johnson's algorithm is $O\big((|\mathcal{V}| + |\mathcal{A}|)(c+1)\big)$. Furthermore, if the weight of each arrow is bounded and $\mathcal{V}$ is finite, then (P3) implies (P2).

6. Abstractions to Reduce the Graph Size

A problem that arises when applying CTG–based analysis is that the set of vertices $\mathcal{V}$ is often infinite. In this section, we introduce results that allow for reducing an infinite CTG into a finite graph while preserving relevant properties of the graph. Such a reduction is called an *abstraction*. Previous work has used abstractions to reduce the infinite state space of timed processes [**?**] and timed hybrid automatons [**?**] into a finite number of states, allowing for algorithmic analysis.

Our general approach is to cover $\mathbb{S}_0^{n-1}$ with a finite number of sets, which we use as replacements for individual points as vertices in graphs. Given a set S, a *cover* of S is a collection of sets $\{P^i\}_{i\in\mathcal{I}}$ indexed over $\mathcal{I} \subset \mathbb{N}$ such that $P^i \subset S$ for each $i \in \mathcal{I}$, and

$$S = \bigcup_{i\in\mathcal{I}} P^i.$$

Given a conical hybrid system $\mathcal{H} := (C, f, D, G)$ with modes $\mathcal{Q}$, we consider a cover of $\mathbb{S}_0^{n-1}$ for each mode. That is, for each $q \in \mathcal{Q}$, let $\mathcal{P}_q := \{P_q^i\}_{i\in\mathcal{I}_q}$ be a cover of $\mathbb{S}_0^{n-1}$ with index set $\mathcal{I}_q$. We impose that $\mathcal{P}_q$ is a finite collection of sets to allow for computational tractability, we write index sets in the form $\mathcal{I}_q := \{0, 1, \ldots, m\}$ where $m \in \mathbb{N}$. For each $e := (q^\ominus, q^\oplus) \in \mathcal{E}$, let

$$\mathcal{I}_e^\ominus := \Big\{ i \in \mathcal{I}_e \;\Big|\; P_{q^\ominus}^i \cap D_e \neq \varnothing \Big\} \quad \text{and} \quad \mathcal{I}_e^\oplus := \Big\{ i \in \mathcal{I}_e \;\Big|\; P_{q^\oplus}^i \cap (A_e D_e) \neq \varnothing \Big\}. \tag{36}$$

Thus, $z \in D_e$ and $e := (q^\ominus, q^\oplus) \in \mathcal{E}$,

$$\exists i^\ominus \in \mathcal{I}_e^\ominus : z \in P_{q^\ominus}^{i^\ominus} \quad \text{and} \quad \exists i^\oplus \in \mathcal{I}_e^\oplus : A_e z \in P_{q^\oplus}^{i^\oplus}. \tag{37}$$

In other words, for each $e := (q^\ominus, q^\oplus) \in \mathcal{E}$,

$$D_e \subset \bigcup_{i^\ominus \in \mathcal{I}_e^\ominus} P_{q^\ominus}^{i^\ominus}, \quad \text{and} \quad A_e D_e \subset \bigcup_{i^\oplus \in \mathcal{I}_e^\oplus} P_{q^\oplus}^{i^\oplus}.$$

We then define abstract conical transition graphs as follows.

Definition 9 (Abstract Conical Transition Graph)**.** Consider a conical hybrid system $\mathcal{H}$ on $\mathbb{R}^n$ with modes $\mathcal{Q}$ and conical transition graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. For each mode $q \in \mathcal{Q}$, let $\mathcal{P}_q = \{P_q^i\}_{i \in \mathcal{I}_q}$ be a cover of $\mathbb{S}_0^{n-1}$ and for each $e \in \mathcal{E}$, let $\mathcal{I}_e^\ominus$ and $\mathcal{I}_e^\oplus$ be defined as in (36). The *abstract conical transition graph* (ACTG) defined by the partitions $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{|\mathcal{Q}|}$ is a directed graph $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{W}})$ with set-valued weights. The vertex set $\widetilde{\mathcal{V}} := \mathcal{V}^\ominus \cup \mathcal{V}^\oplus$ is defined by

$$\mathcal{V}^\ominus := \bigcup_{(q^\ominus, q^\oplus) \in \mathcal{E}} \{q^\ominus\} \times \mathcal{I}_{(q^\ominus, q^\oplus)}^\ominus \qquad \mathcal{V}^\oplus := \bigcup_{(q^\ominus, q^\oplus) \in \mathcal{E}} \{q^\oplus\} \times \mathcal{I}_{(q^\ominus, q^\oplus)}^\oplus. \tag{38}$$

For each $v^\ominus := (q^\ominus, i^\ominus) \in \mathcal{V}^\ominus$ and each $v^\oplus := (q^\oplus, i^\oplus) \in \mathcal{V}^\oplus$, let

$$e := (q^\ominus, q^\oplus), \quad P^\ominus := P_{q^\ominus}^{i^\ominus}, \quad \text{and} \quad P^\oplus := P_{q^\oplus}^{i^\oplus}.$$

There is a *jump arrow* $\mathfrak{a}^{\mathrm{J}} := v^\ominus \xrightarrow{\mathrm{J}} v^\oplus$ in $\widetilde{\mathcal{A}}$ if $(A_e P^\ominus) \cap P^\oplus$ is nonempty. The set-valued weight of $\mathfrak{a}^{\mathrm{J}}$ is

$$\widetilde{\mathcal{W}}(\mathfrak{a}^{\mathrm{J}}) := \big\{ |A_e s^\ominus| \;\big|\; s^\ominus \in P^\ominus \big\}. \tag{39}$$

For each $v^{(0)} := (q, i^{(0)}) \in \mathcal{V}^\oplus$ and each $v^{(\mathrm{f})} := (q, i^{(\mathrm{f})}) \in \mathcal{V}^\ominus$, let $P^{(0)} := P_q^{i^{(0)}}$ and $P^{(\mathrm{f})} := P_q^{i^{(\mathrm{f})}}$. There is a flow arrow $\mathfrak{a}^{\mathrm{F}} := (v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})})$ in $\widetilde{\mathcal{A}}$ if for some $\tau > 0$, there exists $\xi : [0, \tau] \to \mathbb{R}^n$ such that

$$\xi(0) \in P^{(0)} \tag{40a}$$
$$\dot{\xi}(t) = f_q(\xi(t)) \quad \forall t \in (0, \tau) \tag{40b}$$
$$\xi(t) \in C_q \qquad \forall t \in (0, \tau) \tag{40c}$$
$$\operatorname{nrv}(\xi(\tau)) \in P^{(\mathrm{f})}. \tag{40d}$$

The weight of each flow arrow $\mathfrak{a}^{\mathrm{F}} = (q, i^{(0)}) \xrightarrow{\mathrm{F}} (q, i^{(\mathrm{f})})$ is

$$\mathcal{W}(\mathfrak{a}^{\mathrm{F}}) := \big\{ |\xi(\tau)| \;\big|\; \xi : [0, \tau] \to \mathbb{R}^n \text{ satisfies (40) for some } \tau > 0 \big\}. \tag{41}$$

◇

The following result establishes pre-asymptotic stability from ACTG's, analogously to Theorem 2 for CTG's.

Theorem 3. *Let $\mathcal{H} = (C, f, D, G)$ be a conical hybrid system with modes $\mathcal{Q}$ and conical transition graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. For each mode $q \in \mathcal{Q}$, let $\mathcal{P}_q = \{P_q^i\}_{i \in \mathcal{I}_q}$ be a cover of $\mathbb{S}_0^{n-1}$ with $\mathcal{I}_q$ finite, and let $\widetilde{\mathcal{G}} = (\widetilde{\mathcal{V}}, \widetilde{\mathcal{A}}, \widetilde{\mathcal{W}})$ be the abstract conical transition graph defined by $\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_{|\mathcal{Q}|}$. Suppose the following:*

(R1) *For each $q \in \mathcal{Q}$, the origin 0_n is pre-asymptotically stable for (C_q, f_q).*

(R2) *For each arrow $\mathfrak{a} \in \widetilde{\mathcal{A}}$, the weight $\widetilde{\mathcal{W}}(\mathfrak{a})$ is bounded.*

(R3) *For each well-formed elementary cycle w through $\widetilde{\mathcal{G}}$,*

$$\sup \widetilde{\mathcal{W}}(w) < 1.$$

Then, the set $\mathcal{O} := \mathcal{Q} \times \{0_n\}$ is pAS with respect to $\mathcal{H}$.

Proof. The proof proceeds by proving two facts.

Fact 1 There exists $M > 0$ such that for every walk $\tilde{w}$ through $\widetilde{\mathcal{G}}$, we have that

$$\sup \widetilde{\mathcal{W}}(\tilde{w}) \leq M, \tag{42}$$

and if $\tilde{w}$ is infinite, then $\widetilde{\mathcal{W}}(\tilde{w}) = \{0\}$.

Fact 2 For every walk $w := (v_0 \to v_1 \to \cdots \to v_K)$ through $\mathcal{G}$ (for some $K \in \{1, 2, \ldots, \infty\}$), there exists a walk $\tilde{w} := (\tilde{v}_0 \to \tilde{v}_1 \to \cdots \to \tilde{v}_K)$ through $\widetilde{\mathcal{G}}$ such that $\mathcal{W}(w) \subset \widetilde{\mathcal{W}}(\tilde{w})$.

These two facts, along with (R1), imply assumptions (P1)–(P3) of Theorem 2, so we can apply Theorem 2 to conclude $\mathcal{O}$ is pAS.

To prove Fact 1, let $\tilde{w}$ be any walk through $\widetilde{\mathcal{G}}$. Since $|\widetilde{\mathcal{V}}|$ is finite, every walk through $\widetilde{\mathcal{G}}$ returns to a vertex it has already visited every $|\widetilde{\mathcal{V}}| + 1$ or fewer steps (or possibly never, if the length of $\tilde{w}$ is less than $|\widetilde{\mathcal{V}}|$). As a result, $\tilde{w}$ must have the following structure:

1. The walk starts with an acyclical portion consisting of between zero and $|\widetilde{\mathcal{V}}|$-many steps that do not repeat any vertices.
2. The acyclical portion of the walk is followed by any number of cycles (infinitely many, if $\tilde{w}$ is infinite).
3. If the walk is finite, it ends with another acyclical portion of between zero and $|\widetilde{\mathcal{V}}|$-many steps.

Let $\tilde{w}_0$ be the acyclical portion of $\tilde{w}$ before the first cycle, let $\tilde{w}_f$ be the acyclical portion of $\tilde{w}$ after the last cycle, and let $\tilde{w}_c$ be the cyclical part in the middle. Let

$$\mu := \sup\Big\{\sup \widetilde{\mathcal{W}}(\mathfrak{a}') \;\Big|\; \mathfrak{a}' \in \widetilde{\mathcal{A}}\Big\}.$$

By (R2), every step in arrow in $\mathfrak{a}'$ has a bounded weight, so $\mu < \infty$. Thus,

$$\sup \mathcal{W}(\tilde{w}_0) \leq \Big(\sup \widetilde{\mathcal{W}}(\tilde{v}_0 \to \tilde{v}_1)\Big)\Big(\sup \widetilde{\mathcal{W}}(\tilde{v}_1 \to \tilde{v}_2)\Big) \cdots \leq \mu^{|\widetilde{\mathcal{V}}|}.$$

Similarly, $\sup \mathcal{W}(\tilde{w}_f) \leq \mu^{|\widetilde{\mathcal{V}}|}$. For the cyclical portion $\tilde{w}_c$, we have, per (R3), that

$$\sup \widetilde{\mathcal{W}}(\tilde{w}_c) < 1$$

because each cycle multiplies the weight by a value less than 1. Thus, the weight of $\tilde{w}$ must satisfy

$$\sup \mathcal{W}(\tilde{w}) < M := \mu^{2|\widetilde{\mathcal{V}}|},$$

proving Fact 1.

To show Fact 2, let $w := (v_0 \xrightarrow{\ell_0} v_1 \xrightarrow{\ell_1} \ldots \xrightarrow{\ell_{K-1}} v_K)$ be any walk through $\mathcal{G}$ with $K \in \{1, 2, \ldots, \infty\}$. Take any $k \in \{0, 1, \ldots, K-1\}$. Suppose $\ell_k = \text{J}$ and let

$$v^{\ominus} := (q^{\ominus}, s^{\ominus}) := v_k, \quad v^{\oplus} := (q^{\oplus}, s^{\oplus}) := v_{k+1}, \quad \text{and} \quad e := (q^{\ominus}, q^{\oplus}) \in \mathcal{E}.$$

Per (19),

$$s^{\ominus} \in D_e \cap \mathbb{S}_0^{n-1} \quad \text{and} \quad s^{\oplus} = \operatorname{nrv}(A_{(q^{\ominus}, q^{\oplus})} s^{\ominus}) \in A_e D_e \cap \mathbb{S}_0^{n-1}.$$

Because $\{P_{q^{\ominus}}^{i^{\ominus}}\}_{i^{\ominus} \in \mathcal{I}_e^{\ominus}}$ covers D_e, there is some $i^{\ominus} \in \mathcal{I}_e^{\ominus}$ such that $s^{\ominus} \in P^{\ominus} := P_{q^{\ominus}}^{i^{\ominus}}$. Similarly, for some $i^{\oplus} \in \mathcal{I}_e^{\oplus}$, we have $s^{\oplus} \in P^{\oplus} := P_{q^{\oplus}}^{i^{\oplus}}$. Since $s^{\oplus} \in A_e P^{\ominus} \cap P^{\oplus}$, we have that $(q^{\ominus}, i^{\ominus}) \xrightarrow{\text{J}} (q^{\oplus}, i^{\oplus})$ is an arrow in $\widetilde{\mathcal{A}}$. The weight of $v^{\ominus} \xrightarrow{\text{J}} v^{\oplus}$ is $\{|A_e s^{\ominus}|\}$, which is a subset of $\widetilde{\mathcal{W}}(v^{\ominus} \xrightarrow{\text{J}} v^{\oplus})$, per (39).

Alternatively, suppose $\ell_k = \text{F}$ and let

$$v^{(0)} := (q, s^{(0)}) := v_k \quad \text{and} \quad v^{(f)} := (q, s^{(f)}) := v_{k+1}.$$

Take any $r \in \mathcal{W}(v^{(0)} \xrightarrow{\text{F}} v^{(f)})$. Per (22), there exist $\tau > 0$ and $\xi : [0, \tau] \to \mathbb{R}^n$ that satisfy (21) such that $r = |\xi(\tau)|$. We have that $v^{(0)} \in \mathcal{V} \cap G(D)$, so there exists $q^{\ominus} \in \mathcal{Q}$ such that $e := (q^{\ominus}, q) \in \mathcal{E}$, and $s^{\ominus} \in D_e$ such that

$$s^{(0)} = A_e s^{\ominus} \in A_e D_e.$$

Thus, there exists $i^{(0)} \in \mathcal{I}_e^{\oplus}$ such that $s^{(0)} \in P^{(0)} := P_q^{i^{(0)}}$. Similarly, $v^{(f)} \in \mathcal{V} \cap D$, so there exists $q^{\oplus} \in \mathcal{Q}$ such that $e := (q, q^{\oplus}) \in \mathcal{E}$ and

$$s^{(f)} \in D_e.$$

Thus, there exists $i^{(f)} \in \mathcal{I}_e^{\ominus}$ such that $s^{(f)} \in P^{(f)} := P_q^{i^{(f)}}$. We then have that $\tilde{v}^{(0)} := (q, i^{(0)}) \in \mathcal{V}^{\oplus}$ and $\tilde{v}^{(f)} = (q, i^{(f)}) \in \mathcal{V}^{\ominus}$, and

$$\xi(0) = s^{(0)} \in P^{(0)} \quad \text{and} \quad \text{nrv}(\xi(\tau))s^{(f)} \in P^{(f)},$$

satisfying (40). Therefore, $\tilde{v}^{(0)} \xrightarrow{\text{F}} \tilde{v}^{(f)}$ is an arrow in $\widetilde{\mathcal{A}}$ and

$$r = |\xi(\tau)| \in \widetilde{\mathcal{W}}(\tilde{v}^{(0)} \xrightarrow{\text{F}} \tilde{v}^{(f)}).$$

In the manner described above, we construct a walk

$$\tilde{w} := (\tilde{v}_0 \xrightarrow{\ell_0} \tilde{v}_1 \xrightarrow{\ell_1} \dots \xrightarrow{\ell_K} \tilde{v}_K),$$

and since

$$\mathcal{W}(v_k \xrightarrow{\ell_k} v_{k+1}) \subset \widetilde{\mathcal{W}}(\tilde{v}_k \xrightarrow{\ell_k} \tilde{v}_{k+1}) \quad \forall k \in \{0, 1, \dots, K-1\}$$

we have that

$$\mathcal{W}(w) \subset \widetilde{\mathcal{W}}(\tilde{w}),$$

completing the proof of Fact 2.

It follows from Facts 1 and 2 that (P2) and (R3) hold, so by Theorem 2, the set $\mathcal{O}$ is pAS for $\mathcal{H}$. □

7. Numerical Example

In this section, we present an example where we construct an abstract CTG for a hybrid system with modes and apply Theorem 3 to determine asymptotic stability of the origin. In particular, we consider a hybrid system $\mathcal{H}$ as in (10) in $\mathbb{R}^2$ with two modes, $\mathcal{Q} := \{0, 1\}$. The system has linear flows maps in each mode $q \in \mathcal{Q}$ defined by $\dot{z} = A_q z$ where,

$$A_0 = \begin{bmatrix} 2 & 2 \\ -3 & 1 \end{bmatrix} \qquad A_1 = \begin{bmatrix} -1 & 1 \\ -4 & -2 \end{bmatrix}.$$

The eigenvalues of A_0 and A_1 are complex, resulting in flows that spiral around the origin, with the flows in mode $q = 0$ spiraling outward and the flows in $q = 1$ spiraling inward. The components of the flow set in each mode are

$$C_0 := \{(x_1, x_2) \in \mathbb{R}^2 \mid x_1 \leq 0\} \quad \text{and} \quad C_1 := \mathbb{R}^2,$$

where the choice of $C_0 \neq \mathbb{R}^2$ is important to ensure that the origin is stable for flows in mode 0, since solutions spiral outward but can only flow for a finite amount of time before reaching the boundary of C_0.

In each mode, the system can jump within the same mode or jump to the other mode, so the set of mode transition edges is

$$\mathcal{E} := \{(0, 0), (0, 1), (1, 0), (1, 1)\}.$$

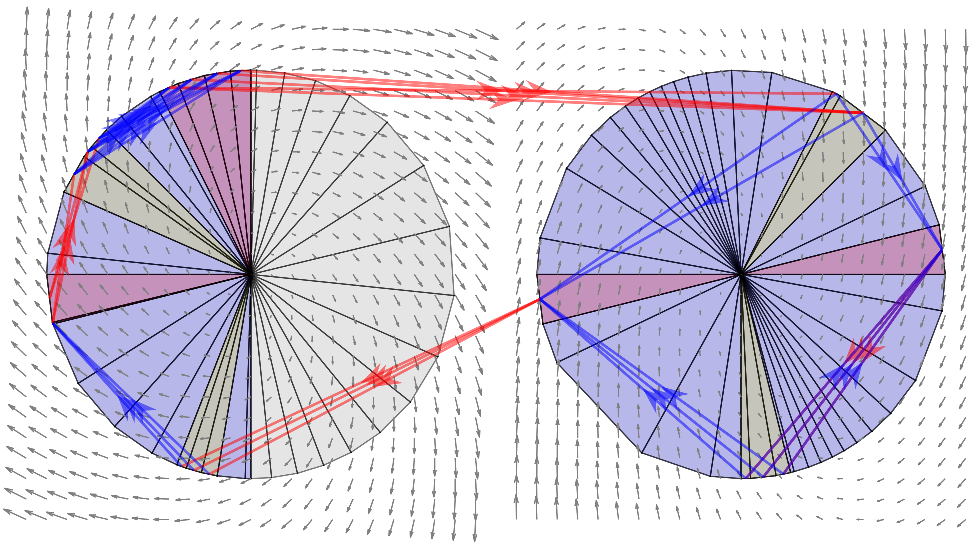

Figure 5: The system $\mathcal{H}$ from Section 7 overlaid by an ACTG. Mode $q = 0$ is shown on the left and $q = 1$ is on the right. For each $q \in \mathcal{Q}$ and $e \in \mathcal{E}$, the set C_q is blue, D_e is red, and $A_e D_e$ are yellow. Jump arrows are drawn as red lines and flow arrows are blue.

The jump map for each transition $e \in \mathcal{E}$ is defined by a linear map $z^+ = A_e z$, where

$$\begin{aligned} A_{(0,0)} &= \begin{bmatrix} 1 & 1/2 \\ -2 & 2 \end{bmatrix}, & A_{(0,1)} &= \gamma \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \\ A_{(1,0)} &= \begin{bmatrix} 1 & 3 \\ 4 & 2 \end{bmatrix}, & A_{(1,1)} &= \begin{bmatrix} 0 & 1 \\ 0 & -1 \end{bmatrix}, \end{aligned} \tag{43}$$

where $\gamma > 0$ is a parameter we discuss in Section 7.1. The jump sets to trigger a jump along each transition are

$$\begin{aligned} D_{(0,0)} &:= \operatorname{cone}\left(\left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} -4 \\ -1 \end{smallmatrix}\right]\right), & D_{(0,1)} &:= \operatorname{cone}\left(\left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 2 \end{smallmatrix}\right]\right), \\ D_{(1,0)} &:= \operatorname{cone}\left(\left[\begin{smallmatrix} -4 \\ -1 \end{smallmatrix}\right], \left[\begin{smallmatrix} -1 \\ 0 \end{smallmatrix}\right]\right), & D_{(1,1)} &:= \operatorname{cone}\left(\left[\begin{smallmatrix} 1 \\ 0 \end{smallmatrix}\right], \left[\begin{smallmatrix} 4 \\ 1 \end{smallmatrix}\right]\right). \end{aligned}$$

A plot of the sets in $\mathcal{H}$ is shown in Figure 5, overlaid with the arrows of the conical transition graph.

The conical partition $\mathcal{P}_0$ and $\mathcal{P}_1$ are constructed using a method similarly to the authors of [14], with additional partitions added as needed so that each boundary of C_q, D_e, and

$A_e D_e$ align with the boundaries of cones in the partition. As a result, every cone in the partition is either entirely inside or entirely outside C_q, D_e, and $A_e D_e$, respectively.

The construction of flow arrows requires determining reachability from each cone in $G(D)$ to each cone in D via flows in C. In each cone of the conical partition, we find the adjacent cones (those that share a boundary), and determine the direction of flow through the boundary. In $\mathbb{R}^2$, all convex cones are polyhedral, which we exploit in our implementation. The find the full set of reachable points reachability analysis, we over approximate the reachable set within a cone K from a polyhedron set of initial positions $P_0 \subset K$ for flows along $\dot{z} = A_q z$ using the fact that $A_q z \in A_q K$ for all $x \in K$. Thus, the reachable set from P_0 in K is given by $(P_0 + A_q K) \cap K$. By picking $P_0 \supset \mathbb{S}^n$, we can over approximate the change in magnitude of a solution as it flows through a cone, allowing us to construct the weights of flow arrows. The code for this example is available at github.com/pwintz/conical-transition-graph.

7.1. Results

In Figure 6, we present the maximum and minimum weights for cycles through the ACTG for various choices of $\gamma > 0$, used to define $A_{(0,1)}$ in (43). We see that for small values of γ, the maximum weight is less than 1, satisfying (R3) in Theorem 3. Furthermore, (R1) and (R2) can also be shown to hold. Therefore, by Theorem 3, the set $\mathcal{Q} \times \{0_n\}$ is pAS for $\mathcal{H}$. Increasing γ above $\gamma \approx 10^{-1}$, however, causes the maximum cycle weight to become greater than 1, so Theorem 3 no longer applies. Note, however, that this is insufficient to conclude that the system becomes unstable—the test is indeterminate and the actual value of γ where instability occurs is likely larger. Over approximations used in the construction of the ACTG cause the maximum walk weight to be inflated. Examining Figure 6, we see that the effect of modifying γ becomes saturated. As γ increases, the minimum cycle weight increases up to a point. After $\gamma = 10^0$, increasing γ has no effect on the minimum cycle weight. The cause of this is the presence of cycles in the graph that don't pass through the transition that depends on γ. Similarly, as γ decreases toward zero, the maximum cycle weight also saturates, as the cycle with the largest cycle becomes one with no dependence on γ.

8. Future Work

There are several avenues for future work on conical transition graphs. There are some promising directions for expanding the generality of the proposed approach. One could relax assumptions on the system to allow for more general types of dynamics, such as allowing for higher order homogenous systems. In particular, the approach is agnostic to how quickly the magnitude of the flow map grows along each ray, so long as all the flows along each ray points in the same direction. In fact, that requirement could also be relaxed to allow for systems with moderate nonlinearities in the direction of flow. We are also interested in extending the CTG approach to include a broader class of hybrid systems, in particular hybrid systems with set-valued flow and jump maps as in [2, Thm. 3.16]. By extending the approach to allow for set-valued dynamics, could

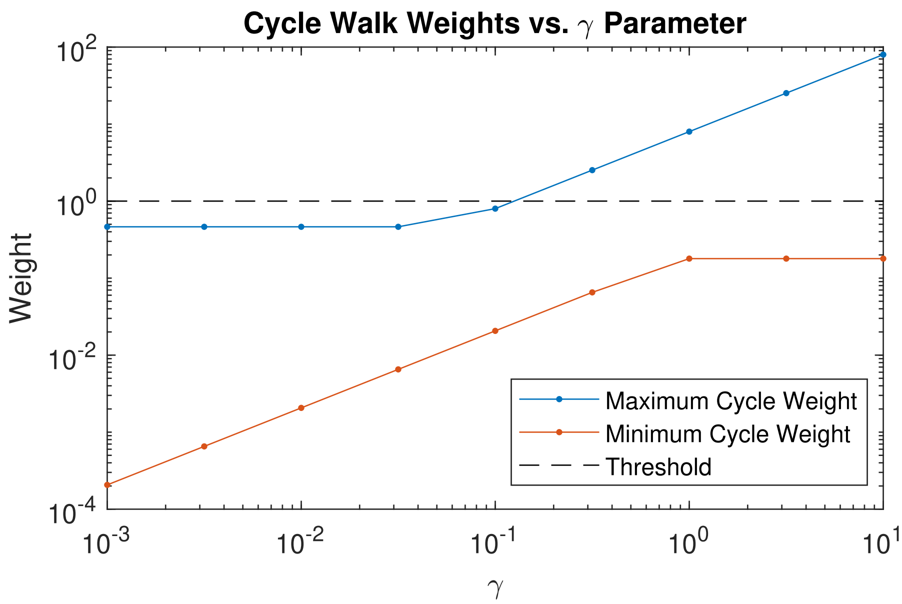

Figure 6: Maximum and minimum weights of cycles in the ACTG of the hybrid system $\mathcal{H}$ described in Section 7 for various values of γ.

over approximate nonlinear vector field as a set-valued map that contains all the flow directions along a particular ray from the origin, or within some cone.

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Appendix A. Additional Results and Proofs

This section contains results omitted from article 1.

Lemma 4. *Let $\mathcal{H}$ be a conical hybrid system with constant flows and let ϕ be any solution to $\mathcal{H}$. For all $r > 0$, the hybrid arc $\psi(t, j) := r\phi(t/r, j)$ for all $(t, j) \in \operatorname{dom}(\psi) := \{(t, j) \mid (t/r, j) \in \operatorname{dom}(\phi)\}$ is a solution to $\mathcal{H}$.*

Proof. Let $\mathcal{H} = (C, f, D, G)$ be a conical hybrid system with constant flows, let ϕ be any solution to $\mathcal{H}$, and for any $r > 0$, let $\psi(t, j) := r\phi(t/r, j)$ for all $(t, j) \in \operatorname{dom}(\psi) := \{(t, j) \mid (t/r, j) \in \operatorname{dom}(\phi)\}$.

We have that t_j is a jump time in $\operatorname{dom}(\psi)$ if and only if t_j/r is a jump time in $\operatorname{dom}(\phi)$, so $\phi(t_j/r, j-1) \in D$. Since D is a cone, $\psi(t_j, j-1) = r\phi(t_j/r, j-1)$ is also in D. By the linearity of G,

$$\begin{aligned} G(\psi(t_j, j-1)) &= G(r\phi(t_j/r, j-1)) \\ &= rG(\phi(t_j/r, j-1)) \\ &= r\phi(t_j/r, j) \\ &= \psi(t_j, j), \end{aligned}$$

so ψ satisfies the jump conditions.

Take any pair of consecutive jump times t_j and t_{j+1} in $\operatorname{dom}(\psi)$ such that $t_{j+1} > t_j$, meaning $I := [t_j, t_{j+1}]$ is an interval of flow in $\operatorname{dom}(\psi)$. Then, $[t_j/r, t_{j+1}/r]$ is also an interval of flow in $\operatorname{dom}(\phi)$. For each $t \in I$, we have that $\psi(t, j) = r\phi(t/r, j) \in C$ because C is a closed cone. From flow condition (9) in the definition of hybrid solutions, we have that $\dot{\phi}(t, j) = f(\phi(t, j))$ for almost all $t \in I$. Thus, using the chain rule and the fact that f is constant-valued, we find that

$$\begin{aligned} \dot{\psi}(t, j) &= \frac{d}{dt}\Big(r\dot{\phi}(t/r, j)\Big) \\ &= r\left(\frac{d}{dt}\bigg|_{t=t/r} \phi(t, j)\right)\left(\frac{d}{dt}\bigg|_{t} (t/r)\right) \\ &= rf(\phi(t/r, j))(1/r) \\ &= f(\psi(t, j)) \end{aligned}$$

for almost all $t \in I$, so ψ satisfies the flow conditions in the definition of a hybrid solution. Therefore, ψ is a solution to $\mathcal{H}$. □

Lemma 5. *Let $\mathcal{H}$ be a conical hybrid system with linear flows and let ϕ be any solution to $\mathcal{H}$. For all $r > 0$, the hybrid arc ψ defined by $\psi(t, j) := r\phi(t, j)$ for all $(t, j) \in \mathrm{dom}(\psi) := \mathrm{dom}(\phi)$ is a solution to $\mathcal{H}$.*

Proof. Let $\mathcal{H} = (C, f, D, G)$ be a conical hybrid system with linear flows, let ϕ be any solution to $\mathcal{H}$, and for any $r > 0$, let $\psi(t, j) := r\phi(t, j)$ for all $(t, j) \in \mathrm{dom}(\psi) := \mathrm{dom}(\phi)$.

For each jump time t_j in $\mathrm{dom}(\phi)$, we have that t_j is a jump time in $\mathrm{dom}(\psi)$ and $\phi(t_j, j - 1) \in D$. Since D is a cone, $\psi(t_j, j - 1) = r\phi(t_j, j - 1)$ is also in D. By the linearity of G,

$$G(\psi(t_j, j - 1)) = G(r\phi(t_j, j - 1)) = rG(\phi(t_j, j - 1)) = r\phi(t_j, j) = \psi(t_j, j),$$

so ψ satisfies the jump conditions.

Take any pair of consecutive jump times t_j and t_{j+1} in $\mathrm{dom}(\phi)$ such that $t_{j+1} > t_j$, meaning $I := [t_j, t_{j+1}]$ is an interval of flow in $\mathrm{dom}(\phi)$, and also in $\mathrm{dom}(\psi)$. For each $t \in I$, we have that $\psi(t, j) = r\phi(t, j) \in C$ because C is a closed cone. Let A be the linear map defining the flow dynamics $\dot{x} = f(x) = Ax$. From flow condition (9) in the definition of hybrid solutions, we have that $\dot{\phi}(t, j) = A\phi(t, j) = f(\phi(t, j))$ for almost all $t \in I$. Thus,

$$\dot{\psi}(t, j) = r\dot{\phi}(t, j) = rA\phi(t, j) = A(r\phi(t, j)) = A\psi(t, j)$$

for almost all $t \in I$, so ψ satisfies the flow conditions in the definition of a hybrid solution. Therefore, ψ is a solution to $\mathcal{H}$. $\square$