Conical Transition Graphs for Analysis of Asymptotic Stability in Hybrid Dynamical Systems

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ADHS, July 1, 2024

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Introduction — Problem Setting

Goal: Develop graph-based analysis of asymptotic stability for *conical hybrid systems*

- Allows determining local asymptotic stability of non-conical hybrid systems by using their *conical approximations*.

Introduction — Graph-based Analysis for Hybrid Systems

Previous work using discrete graphs to analyze hybrid systems:

- Asymptotic stability for
  - switched discrete-time linear systems (Philippe et al., 2016)
  - switched discrete-time nonlinear systems (Kundu and Chatterjee, 2016)
  - switched continuous-time linear systems (Langerak and Polderman, 2005)
- Infinite-horizon reachability for linear hybrid automata (Bogomolov et al., 2017).

The present work is (to the best of our knowledge) the first graph-theoretic approach to analyze *asymptotic stability* in *non-switched* hybrid systems.

Introduction — Hybrid Dynamical Systems Framework

$$\mathcal{H}: \begin{cases} \dot{x} & = f(x) \quad x \in C \\ x^{+} & = g(x) \quad x \in D \end{cases}$$

- flow set $C \subset \mathbb{R}^n$
- flow map $f : C \to \mathbb{R}^n$
- jump set $D \subset \mathbb{R}^n$
- jump map $g : D \to \mathbb{R}^n$

The continuous-time component of $\mathcal{H} = (C, f, D, g)$ is written

$$(C, f).$$

Introduction — Conical Approximations

Definition 1

Given a hybrid system $\mathcal{H} = (C, f, D, g)$ and a point $x_* \in \overline{C} \cup \overline{D}$ such that $g(x_*) = x_*$, the *conical approximation* of $\mathcal{H}$ at x_* is

$$\begin{cases} \check{f}(x) := \text{Constant or linear approximation of } f, & \check{C} := \text{Tangent cone of } C \text{ at } x_*, \\ \check{g}(x) := \text{Linear approximation of } g, & \check{D} := \text{Tangent cone of } D \text{ at } x_*, \end{cases}$$

with each approximation centered at x_*.

Theorem 1 (Goebel and Teel, 2010)

Under sufficient regularity assumptions:

If 0_n is pAS for the conical approximation of $\mathcal{H}$ at x_*, then x_* is pAS for $\mathcal{H}$.

Conical Approximations with Constant Flows

Let $\mathcal{H} = (C, f, D, g)$ be a hybrid system with $x_* \in \overline{C} \cap \overline{D}$ such that

- $g(x_*) = x_*$
- g is continuously differentiable at x_*.
- $f(x_*) \neq 0_n$
- f is continuous at x_*.

Then, the conical approximation of $\mathcal{H}$ at x_* is

$$\breve{\mathcal{H}}: \begin{cases} \dot{x} &= \breve{f}(x) := f(x_*) \text{ [constant]}, \quad \breve{C} := T_C(x_*), \\ x^+ &= \breve{g}(x) := A_{\mathrm{D}} x \text{ [linear]}, \qquad \breve{D} := T_D(x_*), \end{cases}$$

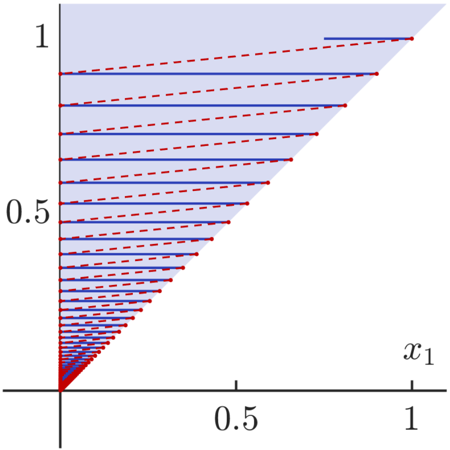
where A_{D} is the Jacobian matrix of g at x_*.

Example 1: Conical Hybrid System

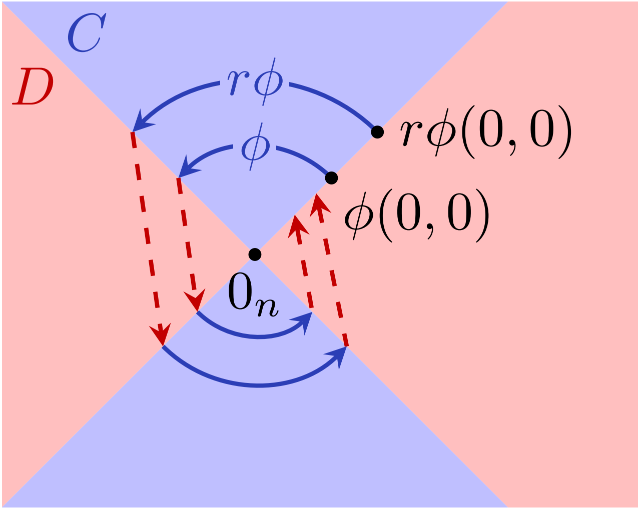
$$\begin{cases} \check{f}(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \forall x \in \check{C} := \{x \in \mathbb{R}^2_{\geq 0} \mid x_2 \geq x_1\}, \\ \check{g}(x) := \begin{bmatrix} 0 \\ \gamma x_1 \end{bmatrix} & \forall x \in \check{D} := \overline{\text{ray}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \end{cases}$$

with $\gamma > 0$.

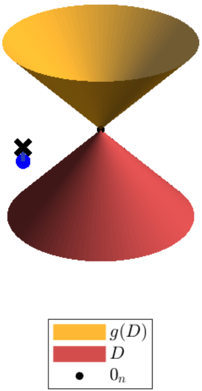
- How to prove the origin asymptotically stable (without a Lyapunov function)?

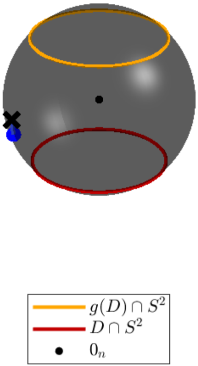


Introduction — Radial Homogeneity



Introduction — Mapping $\mathbb{R}^n$ to Unit Sphere





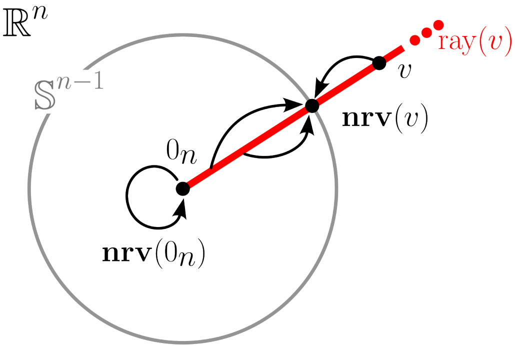
Introduction — Normalized Radial Vectors

The *normalized radial vector* function

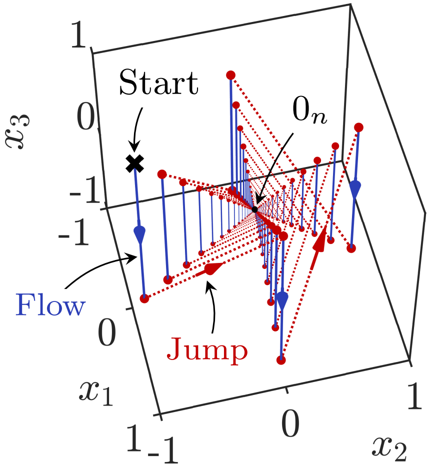
$$\mathbf{nrv} : \mathbb{R}^n \to \mathbb{S}_0^{n-1} := \mathbb{S}^{n-1} \cup \{0_n\}$$

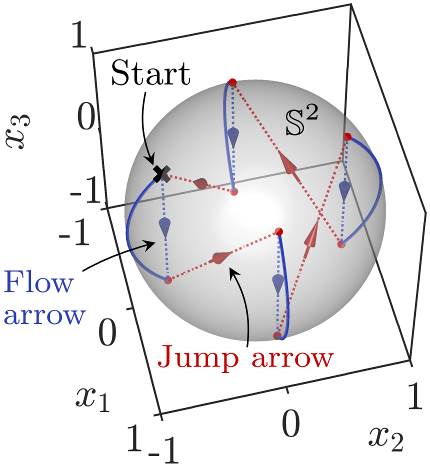
is defined for each $v \in \mathbb{R}^n$ as

$$\mathbf{nrv}(v) := \begin{cases} 0_n, & \text{if } v = 0_n \\ \dfrac{v}{|v|}, & \text{if } v \neq 0_n. \end{cases}$$



Introduction — Conical Transition Graph (Sketch)





Introduction — Directed Graphs with Weights

A *directed graph* consists of a set of vertices

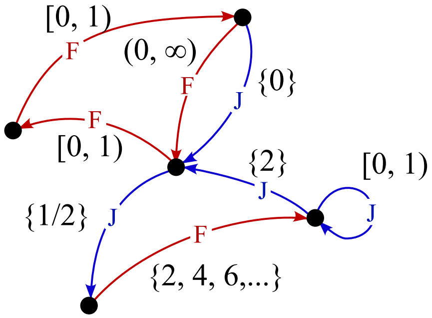
$$\mathcal{V} := \{v_1, v_2, v_3, v_4, v_5\}$$

connected by arrows:

$$\mathcal{A} := \{\mathfrak{a}_1, \mathfrak{a}_2, \mathfrak{a}_3, \text{etc.}\}.$$

Each arrow connects two vertices, e.g.:

$$\mathfrak{a}_1 := v_1 \xrightarrow{\text{F}} v_2, \ \mathfrak{a}_2 = v_3 \xrightarrow{\text{J}} v_1, \ \text{etc.}$$



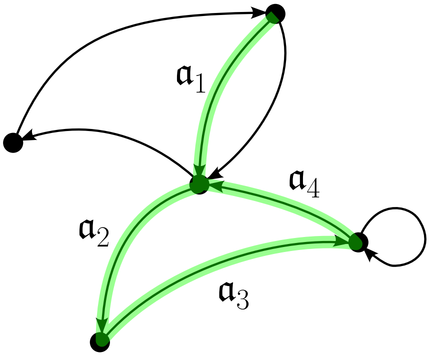
Each arrow is assigned a weight, defined by a *weight function* $\mathcal{W} : \mathcal{V} \rightrightarrows \mathbb{R}$. We write the weight of $\mathfrak{a}$ as $\mathcal{W}(\mathfrak{a}) \subset \mathbb{R}$.

Introduction — Walks Through Graphs

A *walk* w through a graph $\mathcal{G}$ is a sequence of arrows in $\mathcal{A}$:

$$w = (\mathfrak{a}_0,\ \mathfrak{a}_1,\ \ldots,\ \mathfrak{a}_{N-1}) = v_0 \to v_1 \to v_2 \to \cdots \to v_N,$$

such that $\mathfrak{a}_i = v_i \to v_{i+1}$ for each $i = 0, 1, \ldots, N-1$.



Introduction — Walk Weights

The *set-valued weight* of a finite-length walk $w = (\mathfrak{a}_0, \mathfrak{a}_2, \dots, \mathfrak{a}_{N-1})$ is defined as the Minkowski product of the weights

$$\mathcal{W}(w) := \left\{ r_0 \cdot r_1 \cdots r_{N-1} \;\middle|\; r_0 \in \mathcal{W}(\mathfrak{a}_0),\; r_1 \in \mathcal{W}(\mathfrak{a}_1),\; \dots,\; r_{N-1} \in \mathcal{W}(\mathfrak{a}_{N-1}) \right\}$$

For an infinite-length walk $w := (\mathfrak{a}_0, \mathfrak{a}_1, \dots)$, we have that $\mathcal{W}(w) = \{0\}$ if and only if

$$\lim_{m \to \infty} \prod_{k=0}^{m} r_k = 0$$

for every sequence $\{r_k\}_{k=0}^{\infty}$ with $r_k \in \mathcal{W}(\mathfrak{a}_k)$ for all $k \in \mathbb{N}$.

Definition: Conical Transition Graph (CTG)

The *conical transition graph* (CTG) of a conical hybrid system $\check{\mathcal{H}} = (\check{C}, \check{f}, \check{D}, \check{g})$ is a weighted, directed graph

$$\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W}).$$

The set of vertices is defined as

$$\mathcal{V} := (\check{D} \cup \check{g}(\check{D})) \cap \mathbb{S}_0^{n-1}$$

For each $v^{(-)} \in \mathcal{V} \cap \check{D}$, there is a *jump arrow* from $v^{(-)}$ to

$$v^{(+)} = \mathbf{nrv}(\check{g}(v^{(-)}))$$

Definition: Conical Transition Graph (CTG) — Flow Arrows

For each

$$v^{(0)} \in \mathcal{V} \cap \check{g}(\check{D}) \quad \text{and} \quad v^{(\mathsf{f})} \in \mathcal{V} \cap \check{D},$$

there is a *flow arrow* from $v^{(0)}$ to $v^{(\mathsf{f})}$ if for some $T > 0$, there exists a function

$$[0, T] \ni t \mapsto \xi(t)$$

that satisfies

$$\begin{aligned} &\xi(0) = v^{(0)} \\ &\dot{\xi}(t) = \check{f}(\xi(t)) \quad && \forall t \in (0, T) \\ &\xi(t) \in \check{C} && \forall t \in (0, T) \\ &\mathbf{nrv}(\xi(T)) = v^{(\mathsf{f})}. \end{aligned} \tag{Flow Arrow ODE}$$

Definition: Conical Transition Graph (CTG) — Weights

The weight of each jump arrow $\mathfrak{a}^{\mathrm{J}} = v^{(-)} \xrightarrow{\mathrm{J}} v^{(+)}$ is

$$\mathcal{W}(\mathfrak{a}^{\mathrm{J}}) := \left\{ |\breve{g}(v^{(-)})| \right\}.$$

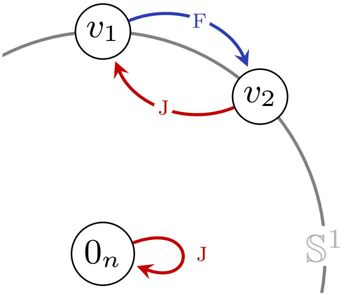
The weight of each flow arrow $\mathfrak{a}^{\mathrm{F}} = v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ is

$$\mathcal{W}(\mathfrak{a}^{\mathrm{F}}) := \left\{ |\xi(T)| \;\middle|\; \xi \text{ satisfies the Flow Arrow ODE for some } T > 0 \right\}.$$

Example 1 (Continued): Construction of CTG

$$\check{\mathcal{H}}\colon \begin{cases} \check{f}(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \forall x \in \check{C} := \left\{ x \in \mathbb{R}^2_{\geq 0} \,\middle|\, x_2 \geq x_1 \right\}, \\ \check{g}(x) := \begin{bmatrix} 0 \\ \gamma x_1 \end{bmatrix} & \forall x \in \check{D} := \overline{\mathrm{ray}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \end{cases}$$

with $\gamma > 0$.



Vertices

$$\mathcal{V} = \{0_n, v_1, v_2\}$$

Arrows

$$\mathcal{A} = \{\underbrace{0_n \xrightarrow{\mathrm{J}} 0_n,\ v_2 \xrightarrow{\mathrm{J}} v_1}_{\text{Jump arrows}},\ \underbrace{v_1 \xrightarrow{\mathrm{F}} v_2}_{\text{Flow arrow}}\}.$$

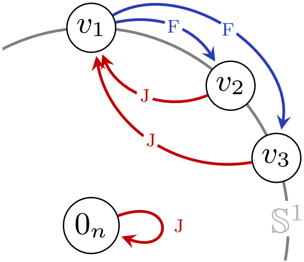
The weights of the arrows are:

$$\mathcal{W}(0_n \xrightarrow{\mathrm{J}} 0_n) = \{0\}$$
$$\mathcal{W}(v_2 \xrightarrow{\mathrm{J}} v_1) = \{\gamma/\sqrt{2}\}$$
$$\mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_2) = \{\sqrt{2}\}.$$

Example 2

Using $\check{f}$ and $\check{g}$ from Example 1, consider the flow and jump sets:

$$\check{C}' := \left\{ x \in \mathbb{R}^2_{\geq 0} \;\middle|\; 2x_2 \geq x_1 \right\}$$
$$\check{D}' := \overline{\mathrm{ray}}\begin{bmatrix} 1 \\ 1 \end{bmatrix} \cup \overline{\mathrm{ray}}\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$



The weights of the arrows are:

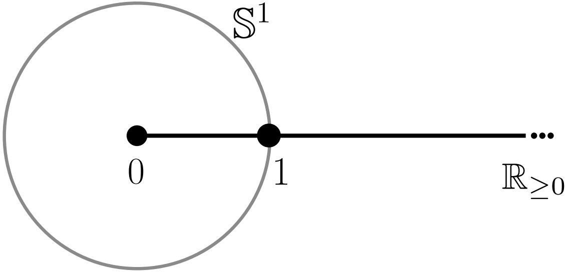
$$\mathcal{W}(0_n \xrightarrow{\mathrm{J}} 0_n) = \{0\}$$
$$\mathcal{W}(v_2 \xrightarrow{\mathrm{J}} v_1) = \{\gamma/\sqrt{2}\} \qquad \mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_2) = \{\sqrt{2}\}$$
$$\mathcal{W}(v_3 \xrightarrow{\mathrm{J}} v_1) = \{2\gamma/\sqrt{5}\} \qquad \mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_3) = \{\sqrt{5}\}.$$

Example 3 — Non-singleton Set-valued Weights

Conical hybrid system:

$$\check{\mathcal{H}} : \begin{cases} \dot{x} &= \check{f}(x) := -1, \quad \check{C} := \mathbb{R}_{\geq 0}, \\ x^{+} &= \check{g}(x) := x/2, \quad \check{D} := \mathbb{R}_{\geq 0}. \end{cases}$$

The origin is asymptotically stable for $\check{\mathcal{H}}$.

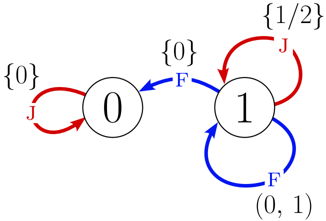


Vertices

$$\mathcal{V} = \{0, 1\}$$

Arrows

$$\mathcal{A} = \left\{0 \xrightarrow{\mathrm{J}} 0,\ 1 \xrightarrow{\mathrm{J}} 1,\ 1 \xrightarrow{\mathrm{F}} 0,\ 1 \xrightarrow{\mathrm{F}} 1\right\}.$$



Main Result

Theorem 2

Let $\breve{\mathcal{H}} = (\breve{C}, \breve{f}, \breve{D}, \breve{g})$ be a conical hybrid system with conical transition graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. Suppose the following:

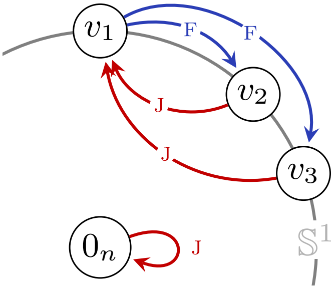
1. *The origin is pre-asymptotically stable for $(\breve{C}, \breve{f})$.*
2. *There exists $M > 0$ such that every walk w through $\mathcal{G}$ satisfies $\sup \mathcal{W}(w) \leq M$.*
3. *Every well-formed infinite-length walk w through $\mathcal{G}$ satisfies $\mathcal{W}(w) = \{0\}$.*

Then, the origin of $\breve{\mathcal{H}}$ is pAS.

When $\mathcal{V}$ is finite:

- Condition 2 is implied by 3 if each individual arrow weight is bounded.
- Condition 3 is satisfied if and only if $\sup \mathcal{W}(w) < 1$ for every well-formed elementary cycle w in $\mathcal{G}$.

Example 2 — Continued



Arrow Weights:

$$\mathcal{W}(0_n \xrightarrow{\text{J}} 0_n) = \{0\}$$
$$\mathcal{W}(v_2 \xrightarrow{\text{J}} v_1) = \{\gamma/\sqrt{2}\} \qquad \mathcal{W}(v_1 \xrightarrow{\text{F}} v_2) = \{\sqrt{2}\}$$
$$\mathcal{W}(v_3 \xrightarrow{\text{J}} v_1) = \{2\gamma/\sqrt{5}\} \quad \mathcal{W}(v_1 \xrightarrow{\text{F}} v_3) = \{\sqrt{5}\}.$$

Weights of Elementary Cycles:

$$\mathcal{W}(0_n \xrightarrow{\text{J}} 0_n) = \{0\}$$
$$\mathcal{W}(v_1 \xrightarrow{\text{F}} v_2 \xrightarrow{\text{J}} v_1) = \mathcal{W}(v_2 \xrightarrow{\text{J}} v_1 \xrightarrow{\text{F}} v_2) = \{\gamma\}$$
$$\mathcal{W}(v_1 \xrightarrow{\text{F}} v_3 \xrightarrow{\text{J}} v_1) = \mathcal{W}(v_3 \xrightarrow{\text{J}} v_1 \xrightarrow{\text{F}} v_3) = \{2\gamma\}$$

$\implies$ For all $0 < \gamma < 1/2$, the origin is pAS.

Future Work

Directions for future work:

1. Expand the scope of systems for which our approach is tractable.
   - Construct CTG's for conical hybrid systems with linear flows.
   - Handle CTG's that have a large or infinite number of vertices.
2. Extend the CTG results to more general hybrid systems:
   - Hybrid systems with switching between logical modes.
   - Hybrid systems with set-valued flow and jump maps.

Questions?

Paper available at
`paulwintz.com/publications`.

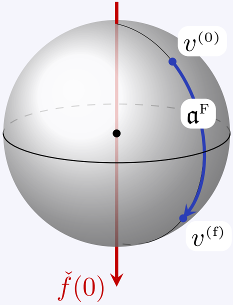
Funding

Proposition 1 (Flow Arrows — Constant Flows)

Let $\check{\mathcal{H}}$ be a conical system with constant flows and let $\mathcal{G}$ be the CTG of $\check{\mathcal{H}}$.

Then, $v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ is a flow arrow in $\mathcal{G}$ from $v^{(0)} \in \mathcal{V} \cap \breve{g}(\check{D})$ to $v^{(\mathrm{f})} \in \mathcal{V} \cap \check{D}$ if and only if

- $\mathbf{nrv}(v_\perp^{(0)}) = \mathbf{nrv}(v_\perp^{(\mathrm{f})})$,
- $v^{(\mathrm{f})} \neq 0_n$ or $v^{(0)} \neq 0_n$,
- $\langle v^{(\mathrm{f})} - v^{(0)},\ \check{f}(0_n)\rangle \geq 0$,
- $v_\perp^{(0)} \neq 0_n \implies \langle v^{(\mathrm{f})} - v^{(0)},\ \check{f}(0_n)\rangle > 0$,
- $\theta v^{(0)} + (1-\theta) v^{(\mathrm{f})} \in \check{C} \quad \forall \theta \in [0, 1]$.



A formula for the weight of $\mathfrak{a}^{\mathrm{F}} := v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ is given in the paper.

Computational Considerations for Finite Conical Transition Graphs

A walk through a graph is called an *elementary cycle* if it starts and ends at the same vertex and does not visit any other vertex more than once.

We can enumerate over all of the elementary cycles using Johnson's enumeration algorithm (Johnson, 1975). The worst-case time complexity of Johnson's algorithm is

$$O\big((\text{no. of vertices} + \text{no. of edges})(\text{no. of elementary cycles} + 1)\big).$$