Conical Transition Graphs for Analysis of Asymptotic Stability in Hybrid Dynamical Systems ⋆

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Abstract: A method is proposed for analyzing asymptotic stability in the conical approximation of a hybrid system. Specifically, this paper introduces the *conical transition graph* (CTG) to simplify the analysis of asymptotic stability in conical approximations by converting solutions to a hybrid system into walks through a discrete graph. By exploiting the fact that pre-asymptotic stability in a conical approximation implies pre-asymptotic stability in the original system, a CTG-based approach can establish asymptotic stability in hybrid systems that have nonlinear flow maps and jump maps without needing to construct a Lyapunov function.

Keywords: Stability and stabilization of hybrid systems, Stability of nonlinear systems

1. INTRODUCTION

For continuous– and discrete–time systems, local asymptotic stability can be determined by linearizing the system and checking the eigenvalues of the resulting Jacobian matrix. For hybrid systems, however, the same ease is currently unavailable. In the conical approximation of a hybrid system, the flow and jumps sets are approximated by tangent cones, and the flow and jump maps are approximated by constant or linear approximations (Goebel et al., 2012, Ch. 9). It was shown in (Goebel and Teel, 2010, Thm. 3.3) that the conical approximation of a hybrid system can be used to determine if a point is pre-asymptotically stable. Namely, if a point is pre-asymptotically stable with respect to the conical approximation, then the center of the approximation in the original hybrid system is locally pre-asymptotically stable. (The prefix "pre-" indicates that some maximal solutions may terminate in finite time due to the solution leaving the region of the state space where it is permitted to evolve.) The utility of (Goebel and Teel, 2010, Thm. 3.3) is currently limited, however, by the fact that it is still generally difficult to show that the origin of a conical approximation is pre-asymptotically stable. The purpose of this paper is to close this gap by introducing the *conical transition graph* (CTG) as a tool to determine asymptotic stability in conical approximations.

While there are limited results for analyzing stability of hybrid systems via conical approximations, there are numerous other approaches for stability analysis in the literature (Tuna and Teel, 2006; Goebel and Sanfelice, 2018) and (Goebel et al., 2012, Thm. 7.30). Discrete graphs [1] have been used to evaluate stability of switched dynamical systems including discrete-time linear systems (Philippe et al., 2016), discrete-time nonlinear systems (Kundu and Chatterjee, 2016), and continuous-time linear systems (Langerak and Polderman, 2005). In contrast to the existing methods for switched systems, the present work is (to the best of our knowledge) the first graph-theoretic approach to analyze asymptotic stability in non-switched hybrid systems (i.e., systems where components of the state vector may range over a continuum at jumps). In the context of reachability analysis, Bogomolov et al. (2017) introduced *conical abstractions* as a graph-based method to compute infinite-horizon reachable sets for linear hybrid automata.

The conical transition graph is designed to simplify the analysis of asymptotic stability of isolated equilibria by creating a simplified representation of ways that solutions to a hybrid system can evolve continuously (called *flows*) or evolve discretely (called *jumps*). Collectively, we refer to flows and jumps as *transitions*. In particular, the CTG is a directed graph with set-valued weights assigned to each arrow. Each vertex in the CTG represents either the origin $0_n \in \mathbb{R}^n$ or a point in the unit sphere $\mathbb{S}^{n-1} \subset \mathbb{R}^n$, where each point $v \in \mathbb{S}^{n-1}$ acts as a representation of all the points in the ray $\{rv \mid r > 0\}$ spanned by v. In this way, we consider the projection of $\mathbb{R}^n$ onto $\mathbb{S}^{n-1} \cup \{0_n\}$, as shown in Fig. 1. Roughly speaking, each arrow in the CTG represents the ways that solutions to a hybrid system, as projected onto $\mathbb{S}^{n-1} \cup \{0_n\}$, can transition (flow or jump) between points in $\mathbb{S}^{n-1} \cup \{0_n\}$. The weight of each arrow contains all possible relative changes in magnitude that a solution can exhibit when it undergoes the transition. Asymptotic stability can be determined from the products

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[1] Throughout, we use *graph* in the sense of *discrete graph*—that is, a set of vertices connected by edges or arrows.

of walks through the CTG. Products converging to zero indicate convergence of solutions to the origin.

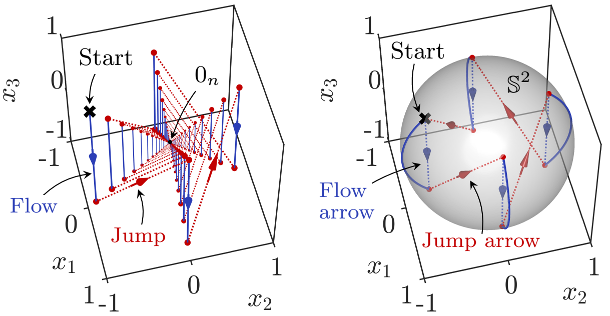


Fig. 1. The evolution of solutions to a hybrid system on $\mathbb{R}^3$ (left) are reduced in the CTG (right) to discrete transitions on $\mathbb{S}^2$, which we label as *flow arrows* and *jump arrows*. In the right image, solid blue curves indicate continuous-time flows projected onto $\mathbb{S}^2$.

2. PRELIMINARIES

For notation, we use $\mathbb{N} := \{0, 1, 2, \dots\}$ and $\mathbb{R}_{\geq 0} := [0, \infty)$. The Euclidean norm of $v \in \mathbb{R}^n$ is written $|v|$. We write the inner product between v_1 and v_2 in $\mathbb{R}^n$ as $\langle v_1, v_2 \rangle$. The zero vector in $\mathbb{R}^n$ is denoted 0_n. The domain of a function f is written $\operatorname{dom} f$. Given a set $S \subset \mathbb{R}^n$, we write the closure as $\overline{S}$. The unit sphere in $\mathbb{R}^n$ is denoted by $\mathbb{S}^{n-1} := \{x \in \mathbb{R}^n \mid |x| = 1\}$, and the unit sphere plus the origin is written as

$$\mathbb{S}_0^{n-1} := \mathbb{S}^{n-1} \cup \{0_n\}. \tag{1}$$

The *normalized radial vector* function $\mathbf{nrv} : \mathbb{R}^n \to \mathbb{S}_0^{n-1}$ is defined for each $v \in \mathbb{R}^n$ as

$$\mathbf{nrv}(v) := \begin{cases} v/|v|, & \text{if } v \neq 0_n \\ 0_n, & \text{if } v = 0_n. \end{cases} \tag{2}$$

2.1 Hybrid Systems

We consider hybrid systems on $\mathbb{R}^n$ written as

$$\mathcal{H} : \begin{cases} \dot{x} \;= f(x), & x \in C, \\ x^+ = g(x), & x \in D, \end{cases} \tag{3}$$

with state $x \in \mathbb{R}^n$, flow set $C \subset \mathbb{R}^n$, flow map $f : C \to \mathbb{R}^n$, jump set $D \subset \mathbb{R}^n$, and jump map $g : D \to \mathbb{R}^n$. The system $\mathcal{H}$ can be written compactly as $\mathcal{H} = (C, f, D, g)$. The continuous-time system formed by removing the discrete dynamics of $\mathcal{H}$ is written as (C, f).

A *hybrid time domain* E is a subset of $\mathbb{R}_{\geq 0} \times \mathbb{N}$ such that for every $(T, J) \in E$, there exists a sequence

$$0 = t_0 \leq t_1 \leq \dots \leq t_{J+1} = T$$

such that

$$\begin{aligned} &E \cap ([0, T] \times \{0, 1, \dots, J\}) \\ &\;= ([t_0,t_1]\times\{0\}) \cup ([t_1,t_2]\times\{1\}) \cup \dots \cup ([t_J,t_{J+1}]\times\{J\}). \end{aligned} \tag{4}$$

Each $t_1, t_2, \dots, t_J$ in (4) is called a *jump time* in E. If $t_{j-1} < t_j$, then $[t_{j-1}, t_j]$ is called an *interval of flow* in E. A function $\phi : \operatorname{dom}\phi \to \mathbb{R}^n$ is called a *hybrid arc* if $\operatorname{dom}\phi$ is a hybrid time domain and ϕ is absolutely continuous on each interval of flow in $\operatorname{dom}\phi$. A hybrid arc ϕ is called *complete* if $\sup\{t + j \mid (t, j) \in \operatorname{dom}\phi\} = \infty$. If a hybrid arc satisfies the dynamics of a hybrid system $\mathcal{H}$, then it is a *solution* of $\mathcal{H}$. See (Goebel et al., 2012; Sanfelice, 2021).

Definition 1. Given a hybrid system $\mathcal{H}$ on $\mathbb{R}^n$, a point $x_* \in \mathbb{R}^n$ is said to be

- *stable* for $\mathcal{H}$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that for every solution ϕ to $\mathcal{H}$ with $|\phi(0, 0) - x_*| \leq \delta$, we have that $|\phi(t, j) - x_*| \leq \varepsilon$ for all $(t, j) \in \operatorname{dom}\phi$.
- *pre-attractive* for $\mathcal{H}$ if there exists $\mu > 0$ such that for each solution ϕ to $\mathcal{H}$ with $|\phi(0, 0) - x_*| \leq \mu$, we have that $(t, j) \mapsto |\phi(t, j) - x_*|$ is bounded and, if ϕ is complete, then
$$\lim_{t+j\to\infty} |\phi(t, j) - x_*| = 0.$$
- *pre-asymptotically stable* (pAS) for $\mathcal{H}$ if x_* is stable and pre-attractive for $\mathcal{H}$.

2.2 Conical Approximations

Let $S \subset \mathbb{R}^n$ be nonempty and let $x \in \overline{S}$. The *contingent cone* $T_S(x)$ is the set of all vectors $v \in \mathbb{R}^n$ such that there exist a sequence of positive real numbers $h_i \to 0^+$ and a sequence of vectors $v_i \to v$ such that $x + h_i v_i \in S$ for all $i \in \mathbb{N}$ (Aubin and Frankowska, 2009). For any $S \subset \mathbb{R}^n$ and $x \in \overline{S}$, the contingent cone of S at x is a *cone*, meaning that for all $x \in T_S(x)$ and all $\alpha \geq 0$, we have that $\alpha x \in T_S(x)$.

For any $x \in \mathbb{R}^n$, we write the open ray from the origin through x as $\operatorname{ray}(x) := \{\alpha x \in \mathbb{R}^n \mid \alpha > 0\}$ and the corresponding closed ray as $\overline{\operatorname{ray}}(x) := \{\alpha x \in \mathbb{R}^n \mid \alpha \geq 0\}$.

The following assumption is necessary for the conical approximation of a hybrid system $\mathcal{H}$ to be well-defined at a point $x_* \in \mathbb{R}^n$.

Assumption 1. For a given hybrid system $\mathcal{H} := (C, f, D, g)$ and $x_* \in \mathbb{R}^n$, suppose that the following conditions hold:

(A1) If $x_* \in \overline{D}$, then $g(x_*) = x_*$ and g is continuously differentiable at x_*.
(A2) If $x_* \in \overline{C}$, then f is continuous at x_*.
(A3) If $x_* \in \overline{C}$ and $f(x_*) = 0_n$, then f is continuously differentiable at x_*.

Definition 2. (Goebel et al. (2012)). Given a hybrid system $\mathcal{H} = (C, f, D, g)$ and a point $x_* \in \mathbb{R}^n$ that satisfy Assumption 1, the *conical approximation* of $\mathcal{H}$ at x_* is

$$\breve{\mathcal{H}}: \begin{cases} \dot{x} \;= \breve{f}(x) := \begin{cases} f(x_*), & \text{if } f(x_*) \neq 0 \\ A_{\mathrm{C}}(x - x_*), & \text{if } f(x_*) = 0, \end{cases} & \breve{C} := T_C(x_*), \\ x^+ = \breve{g}(x) := A_{\mathrm{D}}(x - x_*), & \breve{D} := T_D(x_*), \end{cases} \tag{5}$$

where $A_{\mathrm{C}} := \dfrac{\partial f}{\partial x}(x_*)$ and $A_{\mathrm{D}} := \dfrac{\partial g}{\partial x}(x_*)$ are the Jacobian matrices of g and f at x_*, respectively.

We call a hybrid system $\mathcal{H}$ a *conical hybrid system* if the conical approximation of $\mathcal{H}$ at 0_n is $\mathcal{H}$ itself. Namely, a hybrid system $\breve{\mathcal{H}} = (\breve{C}, \breve{f}, \breve{D}, \breve{g})$ is a conical hybrid system if $\breve{C}$ and $\breve{D}$ are closed cones, $\breve{f}$ is either constant or linear, and $\breve{g}$ is linear. If $\breve{f}$ is constant we say the conical system $\breve{\mathcal{H}}$ has *constant flows* and if $\breve{f}$ is linear, then we say that $\breve{\mathcal{H}}$ has *linear flows*.

An important property of conical hybrid systems is that their dynamics are radially homogenous—that is, a conical hybrid system behaves the same way at all distances from the origin, except for scaling effects.

Proposition 1. Given a conical hybrid system $\breve{\mathcal{H}}$, let ϕ be a solution to $\breve{\mathcal{H}}$ and, for any $r > 0$, let

$$\alpha_r := \begin{cases} 1, & \text{if } \breve{\mathcal{H}} \text{ has linear flows} \\ 1/r, & \text{if } \breve{\mathcal{H}} \text{ has constant flows.} \end{cases} \tag{6}$$

Then, for each $r > 0$, the hybrid arc ψ_r defined by

$$\psi_r(t, j) := r\phi(\alpha_r t, j) \quad \forall(\alpha_r t, j) \in \operatorname{dom} \phi \tag{7}$$

is also a solution to $\breve{\mathcal{H}}$.

The following result establishes local pre-asymptotic stability in a hybrid system via pre-asymptotic stability in its conical approximation.

Theorem 1 (Goebel and Teel (2010), Thm. 3.3). *Suppose a hybrid system $\mathcal{H}$ and a point $x_* \in \mathbb{R}^n$ satisfy Assumption 1. Let $\breve{\mathcal{H}}$ be the conical approximation of $\mathcal{H}$ at x_*. If 0_n is pAS for $\breve{\mathcal{H}}$, then x_* is locally pAS for $\mathcal{H}$.*

2.3 Directed Graphs with Set-valued Weights

This work relies on definitions from graph theory, provided in this section. See (Diestel, 2017) for details.

A *directed graph* $\mathcal{G} = (\mathcal{V}, \mathcal{A})$ consists of a set of *vertices* $\mathcal{V}$ and a set of *arrows* $\mathcal{A}$ that point between vertices. Each arrow in $\mathcal{G}$ starts at some vertex $v_1 \in \mathcal{V}$ and ends at some vertex $v_2 \in \mathcal{V}$. We write an arrow from v_1 to v_2 as $v_1 \to v_2$. In a directed graph, an arrow can have the same start and end point ($v_1 = v_2$), in which case it is called a *loop*.

We also allow for multiple arrows that have the same start and end points. To distinguish between such arrows, we assign each arrow a *label*. An arrow with the label "L" is written as $\mathfrak{a}^{\text{L}} = v_1 \xrightarrow{\text{L}} v_2$. In this work, we use only two labels: "F" and "J," which stand for "flow" and "jump." Thus, for $v_1, v_2 \in \mathcal{V}$, there can be at most two distinct arrows $v_1 \xrightarrow{\text{F}} v_2$ and $v_1 \xrightarrow{\text{J}} v_2$. If the label is irrelevant for a particular point of discussion, then it can be omitted.

A *weighted* directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$ is a directed graph $(\mathcal{V}, \mathcal{A})$ that also includes a weight function $\mathcal{W}$ that defines a weight for each arrow in $\mathcal{A}$. In a typical weighted graph, the weight function assigns a real number to each arrow, but in this work we use *set-valued* weights. Thus, the *weight function* is a set-valued map $\mathcal{W} : \mathcal{A} \rightrightarrows \mathbb{R}$ that maps each arrow $\mathfrak{a}$ in $\mathcal{A}$ to a set of real numbers $\mathcal{W}(\mathfrak{a}) \subset \mathbb{R}$.

Given a graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$, a *walk* w through $\mathcal{G}$ is a sequence of $N \in \{1, 2, \dots\} \cup \{\infty\}$ arrows in $\mathcal{A}$, denoted

$$w = (\mathfrak{a}_0,\ \mathfrak{a}_1,\ \dots,\ \mathfrak{a}_{N-1}) = v_0 \to v_1 \to v_2 \to \cdots \to v_N,$$

such that $\mathfrak{a}_i = v_i \to v_{i+1}$ for each $i = 0, 1, \dots, N-1$.

For a finite-length walk $w = (\mathfrak{a}_0, \mathfrak{a}_2, \dots, \mathfrak{a}_{N-1})$, the *set-valued weight* of w is defined as

$$\mathcal{W}(w) = \left\{ \prod_{k=0}^{N-1} r_k \;\middle|\; \begin{array}{l} r_k \in \mathcal{W}(\mathfrak{a}_k) \\ \quad \forall k = 0, 1, \dots, N-1 \end{array} \right\}. \tag{8}$$

If we let $N = \infty$, then $\mathcal{W}(w)$ may not be well-defined because the infinite product $\prod_{k=0}^{\infty} r_k$ in (8) may not converge. For this paper, however, it is sufficient to define $\mathcal{W}(w)$ if and only if $\prod_{k=0}^{\infty} r_k$ converges to 0 for every choice of $\{r_k\}$. For an infinite-length walk $w := (\mathfrak{a}_1, \mathfrak{a}_2, \dots)$, we have that $\mathcal{W}(w) = \{0\}$ if and only if

$$\lim_{m \to \infty} \prod_{k=0}^{m} r_k = 0 \tag{9}$$

for every sequence $\{r_k\}_{k=0}^{\infty}$ with $r_k \in \mathcal{W}(\mathfrak{a}_k)$ for all $k \in \mathbb{N}$.

For an arrow $\mathfrak{a} \in \mathcal{A}$, we write the *supremal weight* of $\mathfrak{a}$ as $\overline{\mathcal{W}}(\mathfrak{a}) := \sup \mathcal{W}(\mathfrak{a})$. Similarly, for a walk w, we define $\overline{\mathcal{W}}(w) := \sup \mathcal{W}(w)$.

3. CONICAL TRANSITION GRAPH

The CTG is designed to be a simplified representation of a conical hybrid system $\breve{\mathcal{H}}$ to facilitate the analysis of pre-asymptotic stability. To this end, we exploit properties of conical hybrid systems, along with assumptions on the continuous dynamics of the hybrid system, so that the CTG can be used to establish that the origin of $\breve{\mathcal{H}}$ is pAS.

If we consider any ray from the origin and allow every point in the ray to evolve according to the dynamics of $\breve{\mathcal{H}}$, then that ray is (in a sense) preserved by the dynamics. Using this observation, the first simplification in the CTG comes from using the **nrv** function to map $\mathbb{R}^n$ to $\mathbb{S}_0^{n-1}$ so that each point $p \in \mathbb{S}_0^{n-1}$ represents every point in $\operatorname{ray}(p)$.

Mapping $\mathbb{R}^n$ to $\mathbb{S}_0^{n-1}$ reduces the dimension by one and—more importantly—allows for recurrent walks through the CTG despite convergence of solutions (see Fig. 1). For example, suppose that for some $v \in \mathbb{S}^{n-1}$, a solution ϕ to $\breve{\mathcal{H}}$ repeatedly enters $\operatorname{ray}(v)$. That is, $\phi(t_k, j_k) \in \operatorname{ray}(v)$ for a sequence of hybrid times $\{(t_k, j_k)\}$ in $\operatorname{dom} \phi$. Then,

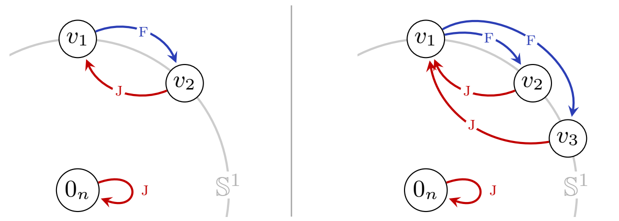
$$v = \mathbf{nrv}(\phi(t_1, j_1)) = \mathbf{nrv}(\phi(t_2, j_2)) = \cdots .$$

Furthermore, the set of possible rays that ϕ can transition into from $\phi(t_k, j_k) \in \operatorname{ray}(v)$ via a single jump or flow is the same at every hybrid time (t_k, j_k) in the sequence. Exploiting this information allows us to uncover patterns in the behavior of $\breve{\mathcal{H}}$.

By collapsing $\mathbb{R}^n$ to $\mathbb{S}_0^{n-1}$, though, we lose information about the magnitude (norm) of solutions. Instead, the weight of each arrow in the CTG typically contains every possible *relative* change of magnitude that a solution $(t, j) \mapsto \phi(t, j)$ can exhibit as $(t, j) \mapsto \mathbf{nrv}(\phi(t, j))$ moves from the arrow's start vertex to its end vertex (both in $\mathbb{S}_0^{n-1}$) via a single jump or a single interval of flow.

The second simplification arising from the CTG is that it allows us to partition the analysis of pre-asymptotic stability by considering separately solutions that are eventually continuous and solutions that are not eventually continuous. A hybrid arc is called *eventually continuous* if it has an interval of flow after the last jump time in its hybrid time domain. The aspects of eventually continuous solutions that are relevant to pre-asymptotic stability in $\breve{\mathcal{H}} = (\breve{C}, \breve{f}, \breve{D}, \breve{g})$ can be determined by analyzing the continuous-time system $(\breve{C}, \breve{f})$. In particular, our results assume that 0_n is pAS for $(\breve{C}, \breve{f})$—which is necessary for 0_n to be pAS for $\breve{\mathcal{H}}$ and can be verified using methods from continuous-time system analysis. Thus, the CTG is a tool for analyzing the behavior of solutions that are not eventually continuous.

Assuming that 0_n is pAS (and thus stable) for $(\breve{C}, \breve{f})$ has the added benefit that if we can show that a given solution converges to 0_n at jump times, then we can establish asymptotic convergence without analyzing the trajectories of solutions *during* intervals of flow. Thus, when determining whether persistently jumping solutions converge to the origin (e.g., to establish pre-asymptotic stability), we can ignore the interior of intervals of flow



(a) CTG for $\breve{\mathcal{H}}$ in Example 1. (b) CTG for $\breve{\mathcal{H}}'$ in Example 2.

Fig. 2. Conical transition graphs from Examples 1 and 2.

and only focus on showing that the solution at jump times converges. By doing so, we treat flows as discrete transitions that take solutions from their values immediately after a jump to their values immediately before the next jump. This effectively ignores the ordinary time required to traverse the flow because it is irrelevant for determining pre-asymptotic stability. Based on this fact, we generalize a flow that takes a solution ϕ from $x^{(0)} \in \mathbb{R}^n$ to $x^{(\mathrm{f})} \in \mathbb{R}^n$ as a flow arrow $\mathbf{nrv}(x^{(0)}) \xrightarrow{\text{F}} \mathbf{nrv}(x^{(\mathrm{f})})$ in the CTG.

We design the CTG as a directed graph with set-valued weights with vertices that live in $\mathbb{S}_0^{n-1}$. Each vector v in $\mathbb{S}_0^{n-1}$ is a vertex in the CTG if it is possible for a solution to $\breve{\mathcal{H}}$ to jump from or to v (i.e., if $v \in \breve{D} \cup \breve{g}(\breve{D})$). An arrow points between vertices v_1 and v_2 in the CTG if a solution to $\breve{\mathcal{H}}$ can move directly from v_1 to $\operatorname{ray}(v_2)$ by a single jump or a single interval of flow. Each arrow is labeled by the type of transition it represents (either flow or jump). The weight of the arrow $v_1 \to v_2$ stores the change in the magnitude of a solution that starts at v_1 and ends in $\operatorname{ray}(v_2)$. By multiplying together the weights of all the arrows in each walk through the CTG, we can analyze the relative change in distance of solutions from the origin.

Definition 3. Let $\breve{\mathcal{H}} = (\breve{C}, \breve{f}, \breve{D}, \breve{g})$ be a conical hybrid system on $\mathbb{R}^n$. Let $\mathcal{L} := \{\text{“J”}, \text{“F”}\}$ be a set of labels, (where J stands for *jump* and F stands for *flow*). The CTG of $\breve{\mathcal{H}}$ is a weighted, directed graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$ where $\mathcal{V} \subset \mathbb{S}_0^{n-1}$ is a set of *vertices*, $\mathcal{A} \subset \mathcal{V}^2 \times \mathcal{L}$ is a set of *arrows* between vertices, and $\mathcal{W} : \mathcal{A} \rightrightarrows \mathbb{R}_{\geq 0}$ is a set-valued *weight function* that assigns a set of nonnegative weights to each arrow. The set of vertices is defined as

$$\mathcal{V} := (\breve{D} \cup \breve{g}(\breve{D})) \cap \mathbb{S}_0^{n-1}. \tag{10}$$

For each $v^{(-)} \in \mathcal{V} \cap \breve{D}$, a *jump arrow* $\mathfrak{a}^{\text{J}} = v^{(-)} \xrightarrow{\text{J}} v^{(+)}$ points from $v^{(-)}$ to

$$v^{(+)} = \mathbf{nrv}(\breve{g}(v^{(-)})) \in \mathcal{V} \cap \breve{g}(\breve{D}). \tag{11}$$

The weight of $\mathfrak{a}^{\text{J}} = v^{(-)} \xrightarrow{\text{J}} v^{(+)}$ is the (singleton) set

$$\mathcal{W}(\mathfrak{a}^{\text{J}}) := \{|\breve{g}(v^{(-)})|\}. \tag{12}$$

There is a *flow arrow* $\mathfrak{a}^{\text{F}} = v^{(0)} \xrightarrow{\text{F}} v^{(\mathrm{f})}$ from $v^{(0)} \in \mathcal{V} \cap \breve{g}(\breve{D})$ to $v^{(\mathrm{f})} \in \mathcal{V} \cap \breve{D}$ if for some $T > 0$, there exists a function $[0, T] \ni t \mapsto \xi(t)$ that satisfies

$$\xi(0) = v^{(0)} \tag{13a}$$

$$\dot{\xi}(t) = \breve{f}(\xi(t)) \quad \forall t \in (0, T) \tag{13b}$$

$$\xi(t) \in \breve{C} \quad \forall t \in (0, T) \tag{13c}$$

$$\mathbf{nrv}(\xi(T)) = v^{(\mathrm{f})}. \tag{13d}$$

The weight of each flow arrow $\mathfrak{a}^{\text{F}} = v^{(0)} \xrightarrow{\text{F}} v^{(\mathrm{f})}$ is

$$\mathcal{W}(\mathfrak{a}^{\text{F}}) := \big\{|\xi(T)| \;\big|\; \xi \text{ satisfies (13) for some } T > 0\big\}. \tag{14}$$

Note that each *vertex* in a CTG is defined as a *vector* in $\mathbb{S}_0^{n-1} \subset \mathbb{R}^n$. We use both terms interchangeably to refer to elements of $\mathcal{V}$, depending on context.

If an arrow $\mathfrak{a} := v_1 \to v_2$ starts at $v_1 \in \mathbb{S}^{n-1}$ (so $v_1 \neq 0_n$), then the weight of $\mathfrak{a}$ is the set of all of the possible *relative* changes in the magnitude of a solution that transitions from $\operatorname{ray}(v_1)$ to $\operatorname{ray}(v_2)$ via a single jump or interval of flow. On the other hand, if $v_1 = 0_n$, then the weight of $\mathfrak{a}$ is the set of all of the possible *absolute* changes in magnitude for a transition from 0_n to $\operatorname{ray}(v_2)$.

The following example demonstrates the construction of the conical transition graph for a conical system.

Example 1. Consider the following conical hybrid system on $\mathbb{R}^2_{\geq 0}$ (the non-negative quadrant of $\mathbb{R}^2$):

$$\breve{\mathcal{H}}\colon \begin{cases} \breve{f}(x) := \begin{bmatrix} 1 \\ 0 \end{bmatrix} & \forall x \in \breve{C} := \{x \in \mathbb{R}^2_{\geq 0} \mid x_2 \geq x_1\}, \\ \breve{g}(x) := \begin{bmatrix} 0 \\ \gamma x_1 \end{bmatrix} & \forall x \in \breve{D} := \overline{\operatorname{ray}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right], \end{cases} \tag{15}$$

with $\gamma > 0$. Let $v_1 := \left[\begin{smallmatrix} 0 \\ 1 \end{smallmatrix}\right]$ and $v_2 := \frac{1}{\sqrt{2}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right]$, so $\breve{g}(\breve{D}) = \overline{\operatorname{ray}}\, v_1$ and $\breve{D} = \overline{\operatorname{ray}}\, v_2$. Thus, the set of vertices is

$$\mathcal{V} = \big(\{0_n\} \cup \operatorname{ray} v_1 \cup \operatorname{ray} v_2\big) \cap \mathbb{S}_0^{n-1} = \{0_n, v_1, v_2\}$$

and the set of arrows is

$$\mathcal{A} = \{\underbrace{0_n \xrightarrow{\text{J}} 0_n,\; v_2 \xrightarrow{\text{J}} v_1}_{\text{Jump arrows}},\; \underbrace{v_1 \xrightarrow{\text{F}} v_2}_{\text{Flow arrow}}\}.$$

The CTG of $\breve{\mathcal{H}}$ is depicted in Fig. 2(a). In Example 4, we continue this example by computing the weights of each arrow and using the CTG to show 0_n is pAS for $\breve{\mathcal{H}}$. ■

The next example considers a modification to the jump set from Example 1 such that solutions are not unique.

Example 2. Let $\breve{\mathcal{H}}' = (\breve{C}', \breve{f}, \breve{D}', \breve{g})$ be a conical hybrid system with $\breve{f}$, $\breve{g}$ given in Example 1, and

$$\breve{C}' := \{x \in \mathbb{R}^2_{\geq 0} \mid 2x_2 \geq x_1\} \quad \text{and} \quad \breve{D}' := \overline{\operatorname{ray}}\left[\begin{smallmatrix} 1 \\ 1 \end{smallmatrix}\right] \cup \overline{\operatorname{ray}}\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right].$$

Define v_1 and v_2 as in Example 1 and let $v_3 := \frac{1}{\sqrt{5}}\left[\begin{smallmatrix} 2 \\ 1 \end{smallmatrix}\right]$. As shown in Fig. 2(b), the set of vertices is $\mathcal{V} = \{0_n, v_1, v_2, v_3\}$ and the set of arrows is

$$\mathcal{A} := \{0_n \xrightarrow{\text{J}} 0_n,\; v_3 \xrightarrow{\text{J}} v_1,\; v_2 \xrightarrow{\text{J}} v_1,\; v_1 \xrightarrow{\text{F}} v_3,\; v_1 \xrightarrow{\text{F}} v_2\}.$$

Note that v_1 has multiple successors and multiple predecessors (the latter is a consequence of $\breve{g}$ being a singular linear map that maps all of $\mathbb{R}^2_{\geq 0}$ onto $\overline{\operatorname{ray}}(v_1)$). This example is continued in Example 5. ■

The need for the weights to be set-valued comes from the fact that there may be multiple solutions to (13) with different final magnitudes, $|\xi(T)|$, as in (14). The following example presents a conical hybrid system with a flow arrow that has a non-singleton weight.

Example 3. Consider the following conical hybrid system:

$$\breve{\mathcal{H}} : \begin{cases} \dot{x} &= \breve{f}(x) := -1, & \breve{C} := \mathbb{R}_{\geq 0}, \\ x^+ &= \breve{g}(x) := x/2, & \breve{D} := \mathbb{R}_{\geq 0}. \end{cases}$$

Every maximal solution to $\breve{\mathcal{H}}$ evolves by a non-deterministic combination of flows and jumps until it reaches 0_n, at which point it must jump from 0_n to 0_n forevermore. Thus, 0_n is pre-asymptotically stable for $\breve{\mathcal{H}}$.

The vertex set of the CTG is $\mathcal{V} = \{0, 1\}$ and the arrow set is $\mathcal{A} = \{0 \xrightarrow{\text{J}} 0, 1 \xrightarrow{\text{J}} 1, 1 \xrightarrow{\text{F}} 0, 1 \xrightarrow{\text{F}} 1\}$. Consider, in particular, the arrow $1 \xrightarrow{\text{F}} 1$. For all $T \in (0, 1)$, the function

$\xi : [0,T] \to \mathbb{R}_{\geq 0}$ defined by $t \mapsto \xi(t) := 1 - t$ satisfies (13) with $v^{(0)} := 1$, $v^{(\mathrm{f})} := 1$, and $|\xi(T)| = 1 - T \in (0,1)$. Thus, $1 \xrightarrow{\mathrm{F}} 1$ is a flow arrow in the CTG with set-valued weight $\mathcal{W}(1 \xrightarrow{\mathrm{F}} 1) = (0,1)$. ■

In addition to having a non-singleton weight, the flow arrow $1 \xrightarrow{\mathrm{F}} 1$ in Example 3 illustrates an exceptional case that we must consider. In Example 3, the origin is pAS for $\check{\mathcal{H}}$, so we want every infinite-length walk through the CTG to have weight $\{0\}$ (see Theorem 2, below). But, the weight of $w := 1 \xrightarrow{\mathrm{F}} 1 \xrightarrow{\mathrm{F}} 1 \xrightarrow{\mathrm{F}} \cdots$ is actually $\mathcal{W}(w) = [0,1)$. To see $\mathcal{W}(w)$ contains $(0,1)$, take any $s > 0$ and let $r_k := \exp(-s/2^{k+1}) \in \mathcal{W}(1 \xrightarrow{\mathrm{F}} 1) = (0,1)$ for each $k \in \mathbb{N}$. Then, by selecting $\{r_k\}_{k=0}^{\infty}$ in (8), we compute

$$\prod_{k=0}^{\infty} r_k = e^{-s(1/2+1/4+1/8+\cdots)} = e^{-s} \in (0,1).$$

Alternatively, selecting $r_k := 1/2 \in \mathcal{W}(1 \xrightarrow{\mathrm{F}} 1)$ results in $\prod_{k=0}^{\infty} 1/2 = 0$. Hence, $\mathcal{W}(w) = [0,1)$. The crux of the problem is that by repeatedly transversing the loop $1 \xrightarrow{\mathrm{F}} 1$, the walk w represents a solution that flows part of the way to the origin, then flows a little more, and a little more, *ad infinitum*, without ever jumping. As indicated by the weight $\mathcal{W}(w)$, we can construct such as sequence of flows that will converge to 0, but also sequences that converge to any value in $[0,1)$. Fortunately, any finite sequence of sequential flow arrows can be replaced by a single flow arrow, whereas any infinite sequence of flow arrows represents a solution that never jumps, so we analyze it using continuous-time methods instead of the CTG. Therefore, we exclude walks with sequential flow arrows from consideration.

Definition 4. (Well-formed Walk). We say that a walk w through a conical transition graph $\mathcal{G}$ is *well-formed* if no pair of sequential arrows in w are both flow arrows.

4. ESTABLISHING PRE-ASYMPTOTIC STABILITY VIA THE CONICAL TRANSITION GRAPH

This section presents a result that allows for pre-asymptotic stability of the origin of a conical hybrid system to be established by analyzing the CTG.

Theorem 2. Let $\check{\mathcal{H}} = (\check{C}, \check{f}, \check{D}, \check{g})$ be a conical hybrid system with conical transition graph $\mathcal{G} = (\mathcal{V}, \mathcal{A}, \mathcal{W})$. Suppose the following:

- (S1) *The origin is pre-asymptotically stable for $(\check{C}, \check{f})$.*
- (S2) *There exists $M > 0$ such that every walk w through $\mathcal{G}$ satisfies $\overline{\mathcal{W}}(w) \leq M$.*
- (S3) *Every well-formed infinite-length walk w through $\mathcal{G}$ satisfies $\mathcal{W}(w) = \{0\}$.*

Then, the origin of $\check{\mathcal{H}}$ is pAS. Moreover, if $\check{\mathcal{H}}$ is the conical approximation of a hybrid system $\mathcal{H}$ about a point $x_* \in \mathbb{R}^n$, then x_* is locally pAS for $\mathcal{H}$.

When $\mathcal{V}$ is finite, condition (S3) is satisfied if and only if $\overline{\mathcal{W}}(w) < 1$ for every elementary cycle w in $\mathcal{G}$. A walk through a graph is called an *elementary cycle* if it starts and ends at the same vertex and does not visit any other vertex more than once. To efficiently check (S3), one can enumerate over all of the elementary cycles using Johnson's enumeration algorithm (Johnson, 1975). For a CTG with $|\mathcal{V}|$ vertices, $|\mathcal{A}|$ arrows, and c elementary circuits (not counting cyclic permutations), the worst-case time complexity of Johnson's algorithm is $O\big((|\mathcal{V}| + |\mathcal{A}|)(c+1)\big)$. Furthermore, if the weight of each arrow is bounded and $\mathcal{V}$ is finite, then (S3) implies (S2).

4.1 Methods for Constructing Conical Transition Graphs

To construct a conical transition graph, one must find the vertex set $\mathcal{V}$, the set of arrows $\mathcal{A}$, and the weight function $\mathcal{W}$. Given $\check{D}$ and $\check{g}$, the vertex set, the jump arrows, and the weights of the jump arrows are immediately available from the definitions in (10)–(12). The main difficulty in constructing the CTG for a given conical hybrid system is finding the flow arrows and their weights.

Computing flow arrows is straightforward for a conical hybrid system $\check{\mathcal{H}}$ with constant flows, due to the simple form of solutions to $\dot{x} = f(0)$ from x_0, namely $t \mapsto x_0 + tf(0)$. Thus, we provide explicit formulas for finding flow arrows and their weights for $\check{\mathcal{H}}$ with constant flows.

In the following result, we write the orthogonal component of a vector $v \in \mathbb{R}^n$ from the line spanned by a nonzero vector $x \in \mathbb{R}^n$ as $\mathrm{orth}_x(v) := v - \big(\langle v,\, x\rangle / |x|^2\big)x$.

Proposition 2 (Flow Arrows – Constant Flows). *Let $\check{\mathcal{H}}$ be a conical system with constant flows and $\check{f}(0_n) \neq 0$, and let $\mathcal{G}$ be the CTG of $\check{\mathcal{H}}$. For any vector $v \in \mathbb{R}^n$, we write $v_\perp := \mathrm{orth}_{\check{f}(0_n)}\, v$. For each pair of vectors $v^{(0)} \in \mathcal{V} \cap \check{g}(\check{D})$ and $v^{(\mathrm{f})} \in \mathcal{V} \cap \check{D}$, we have that $v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ is a flow arrow in $\mathcal{G}$ if and only if*

$$\mathbf{nrv}(v_\perp^{(0)}) = \mathbf{nrv}(v_\perp^{(\mathrm{f})}), \tag{16a}$$
$$v^{(\mathrm{f})} \neq 0_n \text{ or } v^{(0)} \neq 0_n, \tag{16b}$$
$$\langle v^{(\mathrm{f})} - v^{(0)},\ \check{f}(0_n)\rangle \geq 0, \tag{16c}$$
$$v_\perp^{(0)} \neq 0_n \implies \langle v^{(\mathrm{f})} - v^{(0)},\ \check{f}(0_n)\rangle > 0, \text{ and} \tag{16d}$$
$$\theta v^{(0)} + (1-\theta) v^{(\mathrm{f})} \in \check{C} \quad \forall \theta \in [0,1]. \tag{16e}$$

The weight of any flow arrow $\mathfrak{a}^{\mathrm{F}} := v^{(0)} \xrightarrow{\mathrm{F}} v^{(\mathrm{f})}$ in $\mathcal{G}$ is

$$\begin{cases} \{0\}, & \text{if } v^{(\mathrm{f})} = 0_n \\ (0,1), & \text{if } v^{(0)} = v^{(\mathrm{f})} = -\mathbf{nrv}\check{f}(0_n) \\ (1,\infty), & \text{if } v^{(0)} = v^{(\mathrm{f})} = \mathbf{nrv}\check{f}(0_n) \\ (0,\infty), & \text{if } v^{(0)} \neq \mathbf{nrv}\check{f}(0_n),\ v^{(\mathrm{f})} = \mathbf{nrv}\check{f}(0_n) \\ \{|v_\perp^{(0)}|/|v_\perp^{(\mathrm{f})}|\}, & \text{otherwise (i.e., } v_\perp^{(0)} \neq 0_n,\ v_\perp^{(\mathrm{f})} \neq 0_n). \end{cases} \tag{17}$$

Equation (16a) implies that

$$v_\perp^{(0)} = 0_n \quad \text{if and only if} \quad v_\perp^{(\mathrm{f})} = 0_n. \tag{18}$$

Furthermore, if $v_\perp^{(0)} \neq 0_n$ and $v_\perp^{(\mathrm{f})} \neq 0_n$, then

$$v_\perp^{(0)}/|v_\perp^{(0)}| = v_\perp^{(\mathrm{f})}/|v_\perp^{(\mathrm{f})}|, \tag{19}$$

as shown in Fig. 3 with $v^* := v_\perp^{(0)}/|v_\perp^{(0)}| = v_\perp^{(\mathrm{f})}/|v_\perp^{(\mathrm{f})}|$.

To illustrate the application of Theorem 2, we return to the conical hybrid system presented in Example 1.

Example 4. Consider the conical hybrid system $\check{\mathcal{H}}$ in Example 1. The weights of the jump arrows can be easily computed from the definition in (12) to be $\mathcal{W}(0_n \xrightarrow{\mathrm{J}} 0_n) = \{0\}$ and $\mathcal{W}(v_2 \xrightarrow{\mathrm{J}} v_1) = \{|\check{g}(v_2)|\} = \{\gamma/\sqrt{2}\}$. By Proposition 2, the weight of the flow arrow is $\mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_2) = \{\sqrt{2}\}$. In Fig. 2(a), we see that each vertex has a single successor, so there are exactly three infinite-length walks through $\mathcal{G}$:

$$w_0 := 0_n \xrightarrow{\mathrm{J}} 0_n \xrightarrow{\mathrm{J}} \cdots, \qquad w_{\mathrm{F}} := v_1 \xrightarrow{\mathrm{F}} v_2 \xrightarrow{\mathrm{J}} v_1 \xrightarrow{\mathrm{F}} \cdots,$$
$$\text{and} \qquad w_{\mathrm{J}} := v_2 \xrightarrow{\mathrm{J}} v_1 \xrightarrow{\mathrm{F}} v_2 \xrightarrow{\mathrm{J}} \cdots.$$

The weights of each infinite-length walk can be directly computed, but to illustrate a method for graphs with

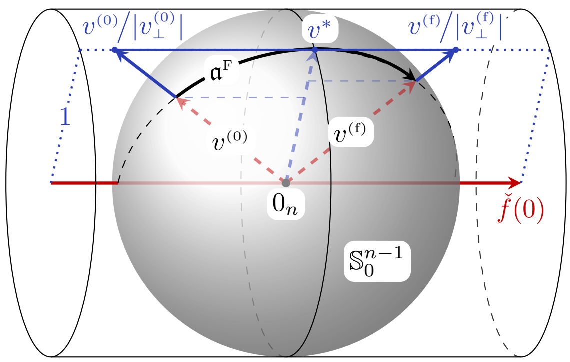


Fig. 3. For a flow arrow $\mathfrak{a}^{\mathrm{F}} := v^{(0)} \xrightarrow{\mathrm{F}} v^{(f)}$, the vector $v^* := v_\perp^{(0)}/|v_\perp^{(0)}|$ is equal to $v_\perp^{(f)}/|v_\perp^{(f)}|$. The vectors v^*, $v^{(0)}/|v_\perp^{(0)}|$, and $v^{(f)}/|v_\perp^{(f)}|$ are colinear on a cylinder of radius 1 with axis $\breve{f}(0_n)$.

infinitely-many infinite-length walks, we will use elementary cycles. There are three elementary cycles in $\mathcal{G}$, namely

$$c_0 := 0_n \xrightarrow{\mathrm{J}} 0_n, \quad c_{\mathrm{F}} := v_1 \xrightarrow{\mathrm{F}} v_2 \xrightarrow{\mathrm{J}} v_1, \quad c_{\mathrm{J}} := v_2 \xrightarrow{\mathrm{J}} v_1 \xrightarrow{\mathrm{F}} v_2.$$

Using the weights computed above, $\mathcal{W}(c_0) = \{0\}$ and

$$\mathcal{W}(c_{\mathrm{F}}) = \mathcal{W}(c_{\mathrm{J}}) = \{(\gamma/\sqrt{2})(\sqrt{2})\} = \{\gamma\}$$

(the equality $\mathcal{W}(c_{\mathrm{F}}) = \mathcal{W}(c_{\mathrm{J}})$ is a result of c_{F} and c_{J} being cyclic permutations of each other). Thus, for all $\gamma \in (0,1)$, every elementary cycle has a supremal weight less than 1, so (S3) is satisfied. For comparison, we can also find the weight of w_{F} without using elementary cycles:

$$\mathcal{W}(w_{\mathrm{F}}) = \{(\sqrt{2})(\gamma/\sqrt{2})(\sqrt{2})(\gamma/\sqrt{2})\cdots\} = \begin{cases}\{0\}, & \text{if } \gamma \in (0,1)\\ \text{undefined}, & \text{if } \gamma \geq 1.\end{cases}$$

Furthermore, it is easily shown that (S1) and (S2) are satisfied. Therefore, by Theorem 2, the point 0_n is pre-asymptotically stable for $\breve{\mathcal{H}}$ when $\gamma \in (0,1)$. ■

Example 5. (Continuation of Example 2). The CTG of $\breve{\mathcal{H}}'$ from Example 2, as shown in Fig. 2(b), has three elementary cycles, not counting cyclic permutations: $c_0 := 0_n \xrightarrow{\mathrm{J}} 0_n$ (with weight $\{0\}$), $c_1 := v_1 \xrightarrow{\mathrm{F}} v_2 \xrightarrow{\mathrm{J}} v_1$ (with weight $\{\gamma\}$, computed in Example 4), and $c_2 := v_1 \xrightarrow{\mathrm{F}} v_3 \xrightarrow{\mathrm{J}} v_1$. From the definition of jump arrow weights in (12), $\mathcal{W}(v_3 \xrightarrow{\mathrm{J}} v_1) = \{2\gamma/\sqrt{5}\}$. Applying Proposition 2, we find $\mathcal{W}(v_1 \xrightarrow{\mathrm{F}} v_3) = \{\sqrt{5}\}$. Thus, $\mathcal{W}(c_2) = \{(2\gamma/\sqrt{5})(\sqrt{5})\} = \{2\gamma\}$. Therefore, if $\gamma < 1/2$, then $\overline{\mathcal{W}}(c_2) < 1$, so we conclude that the origin of $\breve{\mathcal{H}}'$ is pAS for all $\gamma \in (0, 1/2)$. ■

5. FUTURE WORK

Broadly speaking, there are two directions for future work: 1) Expand the scope of systems for which applying Theorem 2 is tractable and 2) extend the CTG results to approximations that are more general than Definition 2.

In the first direction, we highlight two impediments, currently, to applying Theorem 2. In Proposition 2, we gave methods for generating flow arrows and their weights for constant flows, but analogous methods for linear flows remains an open question. Additionally, checking (S2) and (S3) in Theorem 2 is difficult when the CTG contains infinitely-many vertices. The conic abstraction approach from Bogomolov et al. (2017) offers inspiration for solving both impediments. Conic abstractions are constructed by identifying conical regions of the state space where linear flows are approximately "straight." This allows for determining which regions are immediately reachable from each other region, allowing one to create a directed graph that describes which regions are reachable from a given initial set. Similarly, for conical hybrid systems with an infinite CTG, we envision grouping vertices in the CTG to form an "abstracted" CTG that is a finite graph such that satisfaction of (S2) and (S3) by the abstracted CTG implies satisfaction of (S2) and (S3) by the original CTG.

In the second direction for future work, there are two generalizations that can be made to conical approximations (Definition 2) without needing significant modifications to the CTG-based analysis. In particular, conical approximations can be extended to allow for hybrid systems with switching between logical modes as in (Goebel and Teel, 2010, Sec. 7), and for hybrid systems with set-valued flow and jump maps as in (Goebel and Teel, 2010, Thm. 3.16).

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